manuscripta math. 118, 135–149 (2005) © Springer-Verlag 2005

Jianya Liu · Yonghui Wang · Yangbo Ye

A proof of Selberg's orthogonality for automorphic L-functions

Received: 1 September 2004 / Published online: 12 September 2005

Abstract. Let π and π' be automorphic irreducible cuspidal representations of $GL_m(\mathbf{Q_A})$ and $GL_{m'}(\mathbf{Q_A})$, respectively. Assume that π and π' are unitary and at least one of them is self-contragredient. In this article we will give an unconditional proof of an orthogonality for π and π', weighted by the von Mangoldt function $\Lambda(n)$ and $1 - n/x$. We then remove the weighting factor $1 - n/x$ and prove the Selberg orthogonality conjecture for automorphic L-functions $L(s, \pi)$ and $L(s, \pi')$, unconditionally for $m \leq 4$ and $m' \leq 4$, and under the Hypothesis H of Rudnick and Sarnak [20] in other cases. This proof of Selberg's orthogonality removes such an assumption in the computation of superposition distribution of normalized nontrivial zeros of distinct automorphic L-functions by Liu and Ye [12].

Key words. Automorphic L-function – Selberg's orthogonality

1. Introduction

Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbf{Q_A})$, and $s = \sigma + it \in \mathbf{C}$. Then the global L-function attached to π is given by the Euler product of local factors for $\sigma > 1$ (Godement and Jacquet [3]):

$$L(s, \pi) = \prod_p L_p(s, \pi_p),$$

$$\Phi(s, \pi) = L_\infty(s, \pi_\infty) L(s, \pi),$$

J. Liu: Department of Mathematics, Shandong University, Jinan, Shandong 250100, China. e-mail: jyliu@sdu.edu.cn

Y. Wang: Department of Mathematics, Capital Normal University, Beijing 100037, China. e-mail: yhwang@mail.cnu.edu.cn

Y. Ye: Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242-1419, USA. e-mail: yey@math.uiowa.edu

Mathematics Subject Classification (2000): 11F70, 11N05, 11F66, 11M26, 11M41.

Send offprint requests to: Yangbo Ye

Correspondence to: Yangbo Ye

DOI: 10.1007/s00229-005-0563-4

where

$$L_p(s, \pi_p) = \prod_{j=1}^{m} \left(1 - \frac{\alpha_\pi(p, j)}{p^s}\right)^{-1},$$

and

$$L_\infty(s, \pi_\infty) = \prod_{j=1}^{m} \Gamma_{\mathbf{R}}(s + \mu_\pi(j)).$$

Here $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and $\alpha_\pi(p, j)$ and $\mu_\pi(j)$, $j = 1, \dots, m$, are complex numbers associated with π_p and π_∞, respectively, according to the Langlands correspondence. Define

$$a_\pi(p^k) = \sum_{1 \le j \le m} \alpha_\pi(p, j)^k.$$

Note that when $k = 1$, we have $a_\pi(p) = \alpha_\pi(p, 1) + \cdots + \alpha_\pi(p, m)$. If $m = 2$ and π is corresponding to cusp form f, then $a_\pi(p)$ appears as a Fourier coefficient in the Fourier expansion of f.

For $\sigma > 1$, using the Euler product of $L(s, \pi)$, we have

$$\frac{d}{ds} \log L(s, \pi) = -\sum_{n \ge 1} \frac{\Lambda(n) a_\pi(n)}{n^s},$$

where $\Lambda(n)$ is the von Mangoldt function. If π' is an automorphic irreducible cuspidal representation of $GL_{m'}(\mathbf{Q_A})$, we define $L(s, \pi')$, $\alpha_{\pi'}(p, i)$, $\mu_{\pi'}(i)$, and $a_{\pi'}(p^k)$ likewise, for $i = 1, \dots, m'$. If π and π' are equivalent, then $m = m'$ and $\{\alpha_\pi(p, j)\} = \{\alpha_{\pi'}(p, i)\}$ for any p. Hence $a_\pi(n) = a_{\pi'}(n)$ for any $n = p^k$, when $\pi \cong \pi'$.

The Selberg orthogonality conjecture for automorphic L-functions $L(s, \pi)$ was proposed by Selberg [22] in 1989. See also Ram Murty [18] [19].

Selberg's orthogonality conjecture. (i) *For any automorphic irreducible cuspidal representation π of $GL_m(\mathbf{Q_A})$,*

$$\sum_{p \le x} \frac{|a_\pi(p)|^2}{p} = \log\log x + O(1). \tag{1.1}$$

(ii) *For any automorphic irreducible cuspidal representations π of $GL_m(\mathbf{Q_A})$ and π' of $GL_{m'}(\mathbf{Q_A})$,*

$$\sum_{p \le x} \frac{a_\pi(p)\bar{a}_{\pi'}(p)}{p} \ll 1,$$

if π is not equivalent to π'.

The asymptotic formula in (1.1) was proved by Rudnick and Sarnak [20] under a conjecture on convergence of a series on prime powers:

Hypothesis H. *For $k \geq 2$,*

$$\sum_p \frac{|a_\pi(p^k)|^2 \log^2 p}{p^k} < \infty.$$

This Hypothesis H is trivial for $m = 1$, and follows from bounds toward the Ramanujan conjecture for $m = 2$. For $m = 3$ it was proved by Rudnick and Sarnak [20] using the Rankin-Selberg theory, while the case of $m = 4$ was proved by Kim and Sarnak [9]. Thus (1.1) is known unconditionally for $m \leq 4$. For $m > 4$, Hypothesis H is an easy consequence of the Ramanujan conjecture.

Denote $\alpha(g) = |\det(g)|$. In §3, we will prove the following orthogonality.

Theorem 1.1. *Let π and π' be irreducible unitary cuspidal representations of $GL_m(\mathbf{Q_A})$ and $GL_{m'}(\mathbf{Q_A})$, respectively. Assume that at least one of π and π' is self-contragredient: $\pi \cong \tilde{\pi}$ or $\pi' \cong \tilde{\pi}'$. Then*

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \Lambda(n) a_\pi(n) \bar{a}_{\pi'}(n)$$
$$= \begin{cases} \dfrac{x^{1+i\tau_0}}{(1+i\tau_0)(2+i\tau_0)} + O\{x \exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \cong \pi \otimes \alpha^{i\tau_0} \text{ for some } \tau_0 \in \mathbf{R}; \\ O\{x \exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \not\cong \pi \otimes \alpha^{i\tau} \text{ for any } \tau \in \mathbf{R}. \end{cases} \tag{1.2}$$

Here and throughout, c is a positive constant, not necessarily the same at each occurrence.

If $\tau_0 = 0$, i.e. if $\pi \cong \pi'$, then $a_\pi(n) = a_{\pi'}(n)$, and hence Theorem 1.1 states that

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) \Lambda(n) |a_\pi(n)|^2 = \frac{x}{2} + O\{x \exp(-c\sqrt{\log x})\}.$$

Now $\Lambda(n)|a_\pi(n)|^2$ is non-negative. By a classical argument of de la Vallée Poussin, we can remove the weight $1 - n/x$ from Theorem 1.1 when $\pi \cong \pi'$, to get the following prime number theorem for automorphic representations.

Corollary 1.2. *For any self-contragredient automorphic irreducible cuspidal unitary representation π of $GL_m(\mathbf{Q_A})$,*

$$\sum_{n \leq x} \Lambda(n) |a_\pi(n)|^2 = x + O\{x \exp(-c\sqrt{\log x})\}.$$

Without assuming π to be self-contragredient, we can prove

$$\sum_{n \leq x} \Lambda(n) |a_\pi(n)|^2 \sim x \tag{1.3}$$

by the Tauberian theorems of Landau [10] or Ikehara [4]. See the proof of Lemma 5.1 for detail.

In general, we cannot remove the weight $1-n/x$ from Theorem 1.1. But similar to Theorem 1.1, we can establish in Theorem 1.3(i) an asymptotic formula for the expression

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\frac{\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n},$$

in which we are able to remove the weight $1-n/x$ to get Theorem 1.3(ii).

Theorem 1.3. *Let π and π' be given as in Theorem 1.1.*

(i) *We have*

$$\sum_{n\le x}\Big(1-\frac{n}{x}\Big)\frac{\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n} = \begin{cases} \log x + c_1 + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi; \\ \dfrac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)} + c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi\otimes\alpha^{i\tau_0} \text{ for some } \tau_0\in\mathbf{R}^\times; \\ c_2 + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\not\cong\pi\otimes\alpha^{i\tau} \text{ for any } \tau\in\mathbf{R}. \end{cases} \tag{1.4}$$

Here c_1 and c_2 are constants depending on π and π':

$$c_1=\lim_{s\to 0}\Big(-\frac{L'}{L}(s+1,\pi\times\tilde{\pi}')-\frac{1}{s}\Big)-1, \qquad c_2=-\frac{L'}{L}(1,\pi\times\tilde{\pi}').$$

(ii) *We can remove the weight $1-n/x$ in* (i)*, getting*

$$\sum_{n\le x}\frac{\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n} = \begin{cases} \log x + O(1) & \text{if } \pi'\cong\pi; \\ O(1) & \text{if } \pi'\not\cong\pi. \end{cases} \tag{1.5}$$

Note that Theorems 1.1 and 1.3, and Corollary 1.2 are unconditional results. The reason that we can remove $1-n/x$ from (1.4) is that now the main term is of order $\log x$ when $\pi'\cong\pi$, which is substantially bigger than the $O(1)$ of the case of $\pi'\not\cong\pi$. See the proof of Theorem 1.3(ii) in §5 for details.

Corollary 1.4. *Let π and π' be given as in Theorem 1.1. Assume either* (i) *$m\le 4$ and $m'\le 4$, or* (ii) *Hypothesis H. Then*

$$\sum_{p\le x}\frac{a_\pi(p)\bar{a}_{\pi'}(p)\log p}{p} = \begin{cases} \log x + O(1) & \text{if } \pi'\cong\pi; \\ O(1) & \text{if } \pi'\not\cong\pi. \end{cases} \tag{1.6}$$

Corollary 1.5 (Selberg's orthogonality). *Let π and π' be given as in Theorem 1.1. Assume either* (i) *$m\le 4$ and $m'\le 4$, or* (ii) *Hypothesis H. Then*

$$\sum_{p\le x}\frac{a_\pi(p)\bar{a}_{\pi'}(p)}{p} = \begin{cases} \log\log x + O(1) & \text{if } \pi'\cong\pi; \\ O(1) & \text{if } \pi'\not\cong\pi. \end{cases} \tag{1.7}$$

Proofs of Corollaries 1.4 and 1.5 proceed along standard arguments, based on variations of Abel summation. We will thus not give the proofs here, but only point out that Hypothesis H is used to control sums over prime powers in the expression on the left side of (1.5). This way we can obtain a sum taken over primes as in (1.6) and (1.7).

An important application of (1.6) is in the superposition distribution of normalized nontrivial zeros of distinct automorphic L-functions. In Liu and Ye [12] this superposition distribution was computed ([12] Theorem 3.1, pp.421–422) under the assumption of (1.6). Now with Corollary 1.4, we can restate Theorem 3.1 of [12] without assuming Selberg's conjecture: Theorem 3.1 now holds unconditionally for $m \leq 4$, and under Hypothesis H for $m > 4$. This application is indeed our motivation of the present paper.

Under the generalized Ramanujan conjecture, stronger orthogonality relations can be obtained (Liu and Ye [14]):

$$\sum_{n\leq x}(\log n)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n) = \begin{cases} \dfrac{x^{1+i\tau_0}}{1+i\tau_0}\log x - \dfrac{x^{1+i\tau_0}}{(1+i\tau_0)^2} + O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \cong \pi\otimes\alpha^{i\tau_0} \text{ for some } \tau_0\in\mathbf{R}; \\ O\{x\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi' \ncong \pi\otimes\alpha^{i\tau} \text{ for any } \tau\in\mathbf{R}, \end{cases} \tag{1.8}$$

and

$$\sum_{p\leq x}\frac{a_\pi(p)\bar{a}_{\pi'}(p)}{p} = \begin{cases} \log\log x + c_3 + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi; \\ c_4 + \mathrm{Ei}(i\tau_0\log x) + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\cong\pi\otimes\alpha^{i\tau_0} \text{ for some } \tau_0\in\mathbf{R}^\times; \\ c_5 + O\{\exp(-c\sqrt{\log x})\} \\ \qquad \text{if } \pi'\ncong\pi\otimes\alpha^{i\tau} \text{ for any } \tau\in\mathbf{R}, \end{cases} \tag{1.9}$$

where c_3, c_4, and c_5 are constants possibly depending on π and π'. Here Ei is the exponential integral, and

$$\mathrm{Ei}(i\tau_0\log x) = \frac{x^{i\tau_0}}{i\tau_0\log x}\sum_{k=0}^{n}\frac{k!}{(i\tau_0\log x)^k} + O\left(\frac{1}{\log^{n+2}x}\right).$$

It is interesting to compare (1.2) with (1.8), and compare (1.7) with (1.9). Under the Ramanujan conjecture, a remarkable feature of (1.8) is that it is indeed a prime number theorem weighted by Fourier coefficients of automorphic cuspidal representations, while (1.9) describes orthogonality of $a_\pi(n)$ and $a_{\pi'}(n)$ in three cases with different main terms.

Without assuming the Ramanujan conjecture, an unconditional proof of a weighted orthogonality similar to (1.4) was given in Liu and Ye [13]:

$$\sum_{n\leq x}\left(1-\frac{n}{x}\right)\frac{(\log n)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n} \ll \log x,$$

when π and π' are not equivalent. From this weighted orthogonality, the uniqueness of factorization of automorphic L-functions for $GL_m(\mathbf{Q_A})$ was proved in [13].

2. Rankin-Selberg L-functions

We will use the Rankin-Selberg L-functions $L(s, \pi \times \tilde{\pi}')$ as developed by Jacquet, Piatetski-Shapiro, and Shalika [5], Shahidi [23], and Moeglin and Waldspurger [15], where π and π' are automorphic irreducible cuspidal representations of GL_m and $GL_{m'}$, respectively, over $\mathbf{Q}$ with unitary central characters. This finite-part L-function is given by an Euler product of local factors:

$$L(s, \pi \times \tilde{\pi}') = \prod_p L_p(s, \pi_p \times \tilde{\pi}'_p) \tag{2.1}$$

where

$$L_p(s, \pi_p \times \tilde{\pi}'_p) = \prod_{j=1}^{m} \prod_{k=1}^{m'} \left(1 - \frac{\alpha_\pi(p, j)\bar{\alpha}_{\pi'}(p, k)}{p^s}\right)^{-1}.$$

The Archimedean local factor $L_\infty(s, \pi_\infty \times \tilde{\pi}'_\infty)$ is defined by

$$L_\infty(s, \pi_\infty \times \tilde{\pi}'_\infty) = \prod_{j=1}^{m} \prod_{k=1}^{m'} \Gamma_{\mathbf{R}}(s + \mu_{\pi \times \tilde{\pi}'}(j, k))$$

where the complex numbers $\mu_{\pi \times \tilde{\pi}'}(j, k)$ satisfy the trivial bound

$$\mathrm{Re}\, \mu_{\pi \times \tilde{\pi}'}(j, k) > -1. \tag{2.2}$$

Denote

$$\Phi(s, \pi \times \tilde{\pi}') = L_\infty(s, \pi_\infty \times \tilde{\pi}'_\infty) L(s, \pi \times \tilde{\pi}').$$

We will need the following properties of the L-functions $L(s, \pi \times \tilde{\pi}')$ and $\Phi(s, \pi \times \tilde{\pi}')$.

RS1. The Euler product for $L(s, \pi \times \tilde{\pi}')$ in (2.1) converges absolutely for $\sigma > 1$ (Jacquet and Shalika [6]).

RS2. The complete L-function $\Phi(s, \pi \times \tilde{\pi}')$ has an analytic continuation to the entire complex plane and satisfies the functional equation

$$\Phi(s, \pi \times \tilde{\pi}') = \varepsilon(s, \pi \times \tilde{\pi}')\Phi(1 - s, \tilde{\pi} \times \pi')$$

with

$$\varepsilon(s, \pi \times \tilde{\pi}') = \tau(\pi \times \tilde{\pi}') Q_{\pi \times \tilde{\pi}'}^{-s},$$

where $Q_{\pi \times \tilde{\pi}'} > 0$ and $\tau(\pi \times \tilde{\pi}') = \pm Q_{\pi \times \tilde{\pi}'}^{1/2}$ (Shahidi [23], [24], [25], and [26]).

RS3. Denote $\alpha(g) = |\det(g)|$. When $\pi' \not\cong \pi \otimes \alpha^{i\tau}$ for any $\tau \in \mathbf{R}$, $\Phi(s, \pi \times \tilde{\pi}')$ is holomorphic. When $m = m'$ and $\pi' \cong \pi \otimes \alpha^{i\tau_0}$ for some $\tau_0 \in \mathbf{R}$, the only poles

of $\Phi(s, \pi \times \tilde{\pi}')$ are simple poles at $s = i\tau_0$ and $1 + i\tau_0$ coming from $L(s, \pi \times \tilde{\pi}')$ (Jacquet and Shalika [6] and [7], Moeglin and Waldspurger [15]).

RS4. $\Phi(s, \pi \times \tilde{\pi}')$ is meromorphic of order one away from its poles, and bounded in vertical strips (Gelbart and Shahidi [2]).

RS5. $\Phi(s, \pi \times \tilde{\pi}')$ and $L(s, \pi \times \tilde{\pi}')$ are non-zero in $\sigma \geq 1$ (Shahidi [23]).

In addition to the above **RS1**-**RS5**, we will also need to use a region $\mathbf{C}(m, m')$, defined as the complex plane $\mathbf{C}$ with discs

$$|s - 2n + \mu_{\pi\times\tilde{\pi}'}(j,k)| < \frac{1}{8mm'}, \quad n \leq 0, \quad 1 \leq j \leq m, 1 \leq k \leq m'$$

excluded. For $j = 1, \cdots, m$ and $j = 1, \cdots, m'$, denote by $\beta(j,k)$ the fractional part of $\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))$. In addition we let $\beta(0,0) = 0$ and $\beta(m+1, m'+1) = 1$. Then all $\beta(j,k) \in [0,1]$, and hence there exist $\beta(j_1,k_1)$, $\beta(j_2,k_2)$ such that $\beta(j_2,k_2) - \beta(j_1,k_1) \geq 1/(3mm')$ and there is no $\beta(j,k)$ lying between $\beta(j_1,k_1)$ and $\beta(j_2,k_2)$. It follows that the strip $S_0 = \{s : \beta(j_1,k_1) + 1/(8mm') \leq \sigma \leq \beta(j_2,k_2) - 1/(8mm')\}$ is contained in $\mathbf{C}(m,m')$. Consequently, for all $n = 0, -1, -2, \cdots$, the strips

$$S_n = \left\{ s : n + \beta(j_1,k_1) + \frac{1}{8mm'} \leq \sigma \leq n + \beta(j_2,k_2) - \frac{1}{8mm'} \right\} \tag{2.3}$$

are subsets of $\mathbf{C}(m,m')$. This structure of $\mathbf{C}(m,m')$ will be used later.

In [12] and [13], Liu and Ye proved the following lemmas.

Lemma 2.1. *Let $s = \sigma + it$ with $-2 \leq \sigma \leq 2$, $|t| > 2$.*

(i) *Assume $m = m'$ and $\pi' \cong \pi \otimes \alpha^{i\tau_0}$ for some nonzero $\tau_0 \in \mathbf{R}$. If $s \in \mathbf{C}(m,m')$ is not a zero of $L(s, \pi \times \tilde{\pi}')$, then*

$$\begin{aligned} &\frac{d}{ds} \log L(s, \pi \times \tilde{\pi}') \\ &= \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} - \frac{1}{s-1-i\tau_0} - \frac{1}{s-i\tau_0} + O\{\log(Q_{\pi\times\tilde{\pi}'}|t|)\}. \end{aligned}$$

Here and throughout, $\rho = \beta + i\gamma$ denotes a non-trivial zero of $L(s, \pi \times \tilde{\pi}')$.

(ii) *If $\pi' \ncong \pi \otimes \alpha^{i\tau}$ for any $\tau \in \mathbf{R}$, then*

$$\frac{d}{ds} \log L(s, \pi \times \tilde{\pi}') = \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} + O\{\log(Q_{\pi\times\tilde{\pi}'}|t|)\}.$$

Lemma 2.2. (i) *For $|T| > 2$, there exists τ with $T \leq \tau \leq T+1$ such that when $-2 \leq \sigma \leq 2$,*

$$\frac{d}{ds} \log L(\sigma \pm i\tau, \pi \times \tilde{\pi}') \ll \log^2(Q_{\pi\times\tilde{\pi}'}|\tau|).$$

(ii) *If s is in some strip S_n as in* (2.3) *with $n \leq -2$, then*

$$\frac{d}{ds} \log L(s, \pi \times \tilde{\pi}') \ll 1.$$

Furthermore, we need a zero-free region for the Rankin-Selberg L-function $L(s, \pi \times \tilde{\pi}')$ which was proved by Moreno [16] and [17]. See also Gelbart, Lapid, and Sarnak [1], and Sarnak [21].

Lemma 2.3. *Let π and π' be as in Theorem 1.1. Then there are effectively computable constants $c' > 0$ and $c'' \geq 2$ such that $L(s, \pi \times \tilde{\pi}')$ is zero-free in the region*

$$\sigma \geq 1 - \frac{c'}{\log\big(Q_{\pi\times\tilde{\pi}'}(|t| + c'')\big)}, \qquad |t| \gg 1,$$

and at most one exceptional zero in the region

$$\sigma \geq 1 - \frac{c'}{\log\big(Q_{\pi\times\tilde{\pi}'}c''\big)}, \qquad |t| \ll 1.$$

3. Proof of Theorem 1.1

We prove Theorem 1.1 when $\pi' \cong \pi \otimes \alpha^{i\tau_0}$ for some $\tau_0 \in \mathbf{R}$. The proof for case of π and π' being not twisted equivalent is the same with all terms related to τ_0 removed.

By **RS1**, we have for $\sigma > 1$ that

$$J(s) := -\frac{d}{ds}\log L(s, \pi \times \tilde{\pi}') = \sum_{n=1}^{\infty} \frac{\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n^s}.$$

By **RS3** and **RS5**, $J(s)$ is holomorphic in $\sigma > 1$. On $\sigma = 1$, $L(s, \pi \times \tilde{\pi}')$ is nonzero (**RS5**) and has only a simple pole at $s = 1 + i\tau_0$. Thus

$$J(s) = \frac{1}{s - 1 - i\tau_0} + G(s) \tag{3.1}$$

has only a simple pole in $\sigma \geq 1$, where $G(s)$ is analytic for $\sigma \geq 1$. On $\mathbf{C}$, $J(s)$ also has a simple pole at each of the pole at $i\tau_0$, trivial zeros, and nontrivial zeros of $L(s, \pi \times \tilde{\pi}')$.

Note that

$$\frac{1}{2\pi i}\int_{(b)} \frac{y^s}{s(s+1)}ds = \begin{cases} 1 - 1/y & \text{if } y \geq 1, \\ 0 & \text{if } 0 < y < 1, \end{cases}$$

where (b) means the line $\sigma = b > 0$. Then we take $b = 1 + 1/\log x$ and get

$$\begin{aligned}\sum_{n\leq x}\left(1 - \frac{n}{x}\right)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n) &= \frac{1}{2\pi i}\int_{(b)} J(s)\frac{x^s}{s(s+1)}\,ds \\ &= \frac{1}{2\pi i}\left(\int_{b-iT}^{b+iT} + \int_{b-i\infty}^{b-iT} + \int_{b+iT}^{b+i\infty}\right).\end{aligned}$$

The last two integrals are clearly bounded by $\ll \int_T^\infty (x/t^2)\,dt \ll x/T$. Thus,

$$\sum_{n\le x}\left(1-\frac{n}{x}\right)\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}J(s)\frac{x^s}{s(s+1)}\,ds+O\left(\frac{x}{T}\right).$$

Choose a with $-2<a<-1$ such that the line $\sigma=a$ is contained in the strip $S_{-2}\subset \mathbf{C}(m,m')$; this is guaranteed by the structure of $\mathbf{C}(m,m')$. Without loss of generality, let $T>0$ be a large number such that T and $-T$ can be taken as the τ in Lemma 2.2(i). Now we consider the contour

$$\begin{aligned} C_1:&\qquad b\ge\sigma\ge a,\quad t=-T;\\ C_2:&\qquad \sigma=a,\quad -T\le t\le T;\\ C_3:&\qquad a\le\sigma\le b,\quad t=T.\end{aligned}$$

Note that the two poles, some trivial zeros, and certain nontrivial zeros $\rho=\sigma+i\gamma$ of $L(s,\pi\times\tilde{\pi}')$, as well as $s=0,-1$ are passed by the shifting of the contour. The trivial zeros can be determined by the functional equation in **RS2**: $s=-\mu_{\pi\times\tilde{\pi}'}(j,k)$ with $a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1$ and $s=-2-\mu_{\pi\times\tilde{\pi}'}(j,k)$ with $a+2<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1$. Here we have used (2.2) and $-2<a<-1$. Then we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{b-iT}^{b+iT}J(s)\frac{x^s}{s(s+1)}\,ds&=\frac{1}{2\pi i}\left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right)\\ &+\operatorname*{Res}_{s=1+i\tau_0,i\tau_0,0,-1}J(s)\frac{x^s}{s(s+1)}\\ &+\sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1}\operatorname*{Res}_{s=-\mu_{\pi\times\tilde{\pi}'}(j,k)}J(s)\frac{x^s}{s(s+1)}\\ &+\sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1}\operatorname*{Res}_{s=-2-\mu_{\pi\times\tilde{\pi}'}(j,k)}J(s)\frac{x^s}{s(s+1)}\\ &+\sum_{|\gamma|\le T}\operatorname*{Res}_{s=\rho}J(s)\frac{x^s}{s(s+1)}.\end{aligned}\tag{3.2}$$

By Lemma 2.2(i), we get

$$\int_{C_1}\ll\int_a^b\log^2(Q_{\pi\times\tilde{\pi}'}T)\frac{x^\sigma}{T^2}d\sigma\ll\frac{x\log^2(Q_{\pi\times\tilde{\pi}'}T)}{T^2},$$

and the same upper bound also holds for the integral on C_3. By Lemma 2.2(ii), then

$$\int_{C_2}\ll\int_{-T}^{T}\frac{x^a}{(|t|+1)^2}dt\ll\frac{1}{x}.$$

On taking $T\gg x$, the three integrals on C_1, C_2, C_3 are

$$\ll\log^2(Q_{\pi\times\tilde{\pi}'}x).\tag{3.3}$$

If $\tau_0 \in \mathbf{R}^\times$, then

$$\begin{aligned}\operatorname*{Res}_{s=1+i\tau_0, i\tau_0, 0, -1} J(s)\frac{x^s}{s(s+1)} &= \frac{x^{1+i\tau_0}}{(1+i\tau_0)(2+i\tau_0)} + \frac{x^{i\tau_0}}{(i\tau_0)(1+i\tau_0)} \\ &\quad + J(0) - \frac{J(-1)}{x} \\ &= \frac{x^{1+i\tau_0}}{(1+i\tau_0)(2+i\tau_0)} + O(\log x). \end{aligned} \tag{3.4}$$

If $\tau_0 = 0$, then $s = 0$ is a double pole. Computing the residues similarly, we see that the final estimate (3.4) is still true.

Near a trivial zero $s = -\mu_{\pi\times\tilde{\pi}'}(j,k)$ of order l, we can express $J(s)$ as $-l/(s+\mu_{\pi\times\tilde{\pi}'}(j,k))$ plus an analytic function, like in (3.1). The residues at these trivial zeros can therefore be computed similarly to what we have done in (3.4). By (2.2), we know that $-\mathrm{Re}\ \mu_{\pi\times\tilde{\pi}'}(j,k) < 1-\delta$ for some $\delta > 0$. Consequently,

$$\sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1} \operatorname*{Res}_{s=-\mu_{\pi\times\tilde{\pi}'}(j,k)} J(s)\frac{x^s}{s(s+1)} \ll x^{1-\delta}, \tag{3.5}$$

$$\sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1} \operatorname*{Res}_{s=-2-\mu_{\pi\times\tilde{\pi}'}(j,k)} J(s)\frac{x^s}{s(s+1)} \ll x^{-1-\delta}. \tag{3.6}$$

To compute the residues corresponding to nontrivial zeros, we note that $\Phi(s, \pi\times\tilde{\pi}')$ is of order 1 (**RS4**), and $\Phi(1, \pi\times\tilde{\pi}') \neq 0$ (**RS5**), and hence

$$\sum_{\rho} \frac{1}{|\rho(\rho+1)|} < \infty.$$

Consequently,

$$\begin{aligned}\sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} J(s)\frac{x^s}{s(s+1)} &= -\sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho} \frac{1}{s-\rho}\frac{x^s}{s(s+1)} \\ &\ll \sum_{|\gamma|\le T} \left|\frac{x^\rho}{\rho(\rho+1)}\right| \\ &= \Big(\sum_{\substack{|\gamma|\le T\\ \rho\in E}} + \sum_{\substack{|\gamma|\le T\\ \rho\notin E}}\Big)\frac{x^\beta}{|\rho(\rho+1)|}, \end{aligned} \tag{3.7}$$

where E is the set of exceptional zeros in Lemma 2.3(ii). By Lemma 2.3(ii), $|E| \le 1$, and therefore the sum over $\rho \in E$ is clearly $\ll x^{1-\delta}$ for some $\delta > 0$. By Lemma 2.3(i), the sum over $\rho \notin E$ is

$$\ll x\exp\left(-c'\frac{\log x}{\log T}\right)\sum_{\rho}\frac{1}{|\rho(\rho+1)|} \ll x\exp(-c\sqrt{\log x}), \tag{3.8}$$

by taking $T = \exp(\sqrt{\log x}) + d$ for some d with $0 < d < 1$. Hence (3.7) is bounded by $x\exp(-c\sqrt{\log x})$.

Theorem 1.1 then follows from applying (3.3)–(3.6) and (3.8) to (3.2). □

4. Proof of Theorem 1.3 (i)

The proof follows the same line as that of Theorem 1.1, so we only indicate the differences. With the same $J(s)$ and $b = 1/\log x$, we have

$$\sum_{n\le x}\left(1-\frac{n}{x}\right)\frac{\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n} = \frac{1}{2\pi i}\int_{(b)} J(s+1)\frac{x^s}{s(s+1)}\,ds$$
$$= \frac{1}{2\pi i}\left(\int_{b-iT}^{b+iT} + \int_{b-i\infty}^{b-iT} + \int_{b+iT}^{b+i\infty}\right).$$

The last two integrals are clearly bounded by $\ll \int_T^\infty (1/t^2)\,dt \ll 1/T$. Let a and T be as before. Then

$$\sum_{n\le x}\left(1-\frac{n}{x}\right)\frac{\Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)}{n} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} J(s+1)\frac{x^s}{s(s+1)}\,ds + O\left(\frac{1}{T}\right).$$

Define

$$\begin{aligned} D_1 &: \quad b \ge \sigma \ge a, \quad t = -T;\\ D_2 &: \quad \sigma = a, \quad -T \le t \le T;\\ D_3 &: \quad a \le \sigma \le b, \quad t = T. \end{aligned}$$

Note that the two poles, some trivial zeros, and certain nontrivial zeros $\rho = \sigma + i\gamma$ of $L(s+1, \pi\times\tilde{\pi}')$, as well as $s = 0, -1$ are passed by the shifting of the contour. The trivial zeros can be determined by the functional equation in **RS2**: $s+1 = -\mu_{\pi\times\tilde{\pi}'}(j,k)$ with $a < -\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k)) < 1$ and $s+1 = -2-\mu_{\pi\times\tilde{\pi}'}(j,k)$ with $a+2 < -\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k)) < 1$. Then we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{b-iT}^{b+iT} J(s+1)\frac{x^s}{s(s+1)}\,ds\\ &= \frac{1}{2\pi i}\left(\int_{D_1} + \int_{D_2} + \int_{D_3}\right)\\ &\quad + \operatorname*{Res}_{s=0,i\tau_0,-1,-1+i\tau_0} J(s+1)\frac{x^s}{s(s+1)}\\ &\quad + \sum_{a<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1} \operatorname*{Res}_{s=-1-\mu_{\pi\times\tilde{\pi}'}(j,k)} J(s+1)\frac{x^s}{s(s+1)}\\ &\quad + \sum_{a+2<-\mathrm{Re}(\mu_{\pi\times\tilde{\pi}'}(j,k))<1} \operatorname*{Res}_{s=-3-\mu_{\pi\times\tilde{\pi}'}(j,k)} J(s+1)\frac{x^s}{s(s+1)}\\ &\quad + \sum_{|\gamma|\le T} \operatorname*{Res}_{s=\rho-1} J(s+1)\frac{x^s}{s(s+1)}. \end{aligned} \tag{4.1}$$

Similar to the previous argument, we can prove that, except the residues at $0, i\tau_0$, everything on the right of (4.1) is

$$\ll \exp(-c\sqrt{\log x}) \tag{4.2}$$

by taking $T = \exp(\sqrt{\log x}) + d$ for some d with $0 < d < 1$. In particular,

$$\int_{D_1} \ll \int_a^b \log^2(Q_{\pi\times\tilde{\pi}'}T)\frac{x^\sigma}{T^2}d\sigma \ll \frac{\log^2(Q_{\pi\times\tilde{\pi}'}T)}{T^2} \ll \exp(-\sqrt{\log x}).$$

To compute the residues at $0, i\tau_0$, we have to distinguish three cases. If $\pi' \cong \pi$, i.e. $\tau_0 = 0$, then $s = 0$ is a double pole of $J(s+1)x^s/\{s(s+1)\}$, and the residue is

$$\lim_{s\to 0}\frac{d}{ds}\left\{s^2\cdot J(s+1)\frac{x^s}{s(s+1)}\right\} = \log x + \lim_{s\to 0}\Big(J(s+1) - \frac{1}{s}\Big) - 1.$$

If $\pi' \cong \pi \otimes \alpha^{i\tau_0}$ for some $\tau_0 \in \mathbf{R}^\times$, then $s = 0, i\tau_0$ are two different simple poles with residues

$$J(1) + \frac{x^{i\tau_0}}{i\tau_0(1+i\tau_0)}.$$

If $\pi' \not\cong \pi \otimes \alpha^{i\tau}$ for any $\tau \in \mathbf{R}$, then the terms involving τ_0 disappear, and only $s = 0$ is a simple pole, with residue $J(1)$.

Inserting these and (4.2) into (4.1) gives part (i) of Theorem 1.3. □

5. Weight removal

Lemma 5.1. *For any automorphic irreducible cuspidal unitary representation π of $GL_m(\mathbf{Q_A})$, not necessarily self-contragredient, we have*

$$\sum_{n\le x}\Lambda(n)|a_\pi(n)|^2 \sim x.$$

Proof. A Tauberian theorem of Ikehara [4] says that, if $f(s)$ is given for $\sigma > 1$ by a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

with $a_n \ge 0$, and if

$$g(s) = f(s) - \frac{1}{s-1}$$

has analytic continuation to $\sigma \ge 1$, then

$$\sum_{n\le x} a_n \sim x.$$

By **RS1**, **RS3**, and **RS5**, we can apply this theorem to

$$f(s) = -\frac{L'}{L}(s, \pi\times\tilde{\pi}).$$

Lemma 5.1 then follows. □

Proof of Theorem 1.3(ii). We use a technique of Landau [11]. For simplicity, we denote $c(n) = \Lambda(n)a_\pi(n)\bar{a}_{\pi'}(n)/n$, and

$$C(x) = \sum_{n\le x} c(n), \qquad D(x) = \sum_{n\le x}\left(1-\frac{n}{x}\right)c(n).$$

Then

$$\int_0^x C(t)dt = \sum_{n\le x}(x-n)c(n) = xD(x).$$

We begin with an observation that

$$\begin{aligned}\int_x^{x+v} C(t)dt &= (x+v)D(x+v) - xD(x)\\ &= vD(x+v) + x(D(x+v)-D(v)),\end{aligned} \tag{5.1}$$

where $v > 0$ is any real number, but for later use we specify that $v = \sqrt{x}$.

By Theorem 1.3(i), we have

$$vD(x+v) = v\begin{cases}\log x + O(1) & \text{if } \pi' \cong \pi;\\ O(1) & \text{if } \pi' \ncong \pi.\end{cases} \tag{5.2}$$

The last term in (5.1) is

$$\begin{aligned}x(D(x+v)-D(v)) &= x\sum_{n\le x}\left(\frac{n}{x}-\frac{n}{x+v}\right)c(n)\\ &\quad + x\sum_{x<n\le x+v}\left(1-\frac{n}{x+v}\right)c(n)\\ &=: E_1 + E_2,\end{aligned}$$

say.

To estimate E_1 and E_2, we apply Lemma 5.1. It is weaker than Corollary 1.2, but holds for any irreducible unitary cuspidal representations π of $GL_m(\mathbf{Q_A})$, not necessarily self-contragredient. Therefore, we have

$$|E_1| \le \frac{v}{x+v}\sum_{n\le x} n|c(n)| \ll \frac{v}{x+v}\sum_{n\le x}\Lambda(n)(|a_\pi(n)|^2 + |a_{\pi'}(n)|^2) \ll v.$$

On the other hand,

$$\begin{aligned}|E_2| &\le \frac{xv}{x+v}\sum_{x<n\le x+v}|c(n)| \ll \frac{v}{x+v}\sum_{x<n\le x+v} n|c(n)|\\ &\ll \frac{v}{x+v}\sum_{x<n\le x+v}\Lambda(n)(|a_\pi(n)|^2 + |a_{\pi'}(n)|^2) \ll v.\end{aligned} \tag{5.3}$$

Putting (5.2) and (5.3) into (5.1), we get

$$\int_x^{x+v} C(t)dt = v\begin{cases}\log x + O(1) & \text{if } \pi' \cong \pi;\\ O(1) & \text{if } \pi' \ncong \pi.\end{cases} \tag{5.4}$$

Now we consider the difference

$$\begin{aligned}\left|\int_x^{x+v} C(t)dt - vC(x)\right| &= \left|\int_x^{x+v} (C(t)-C(x))dt\right| \ll v \sum_{x<n\le x+v} |c(n)| \\ &\ll \frac{v}{x} \sum_{x<n\le x+v} n|c(n)| \\ &\ll \frac{v}{x} \sum_{x<n\le x+v} \Lambda(n)(|a_\pi(n)|^2 + |a_{\pi'}(n)|^2) \\ &\ll v,\end{aligned}$$

by the same argument as in (5.3). The desired result (1.5) now follows from this and (5.4). □

Acknowledgements. This project is sponsored in part by China NSF Grant #10125101 and a Ministry of Education Major Grant Program in Sciences and Technology (J. Liu), by China NSF Tianyuan Youth Grant #10126001 (Y. Wang), and by the USA National Security Agency under Grant Number MDA904-03-1-0066 (Y. Ye). The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein. The last author wishes to acknowledge support provided by a Mathematical and Physical Sciences Funding Program Award and the Obermann Center for Advanced Studies at the University of Iowa.

References

[1] Gelbart, S.S., Lapid, E.M., Sarnak, P.: A new method for lower bounds of L-functions. C.R. Acad. Sci. Paris, Ser. I **339**, 91–94 (2004)
[2] Gelbart, S.S., Shahidi, F.: Boundedness of automorphic L-functions in vertical strips. J. Amer. Math. Soc. **14**, 79–107 (2001)
[3] Godement, R., Jacquet, H.: Zeta functions of simple algebras. Lecture Notes in Math. **260**, Springer-Verlag, Berlin, 1972
[4] Ikehara, S.: An extension of Landau's theorem in the analytic theory of numbers. J. Math. Phys. M.I.T. **10**, 1–12 (1931)
[5] Jacquet, H., Piatetski-Shapiro, I.I., Shalika, J.: Rankin-Selberg convolutions. Amer. J. Math. **105**, 367–464 (1983)
[6] Jacquet, H., Shalika, J.A.: On Euler products and the classification of automorphic representations I. Amer. J. Math. **103**, 499–558 (1981)
[7] Jacquet, H., Shalika, J.A.: On Euler products and the classification of automorphic representations II. Amer. J. Math. **103**, 777–815 (1981)
[8] Kim, H.: Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. J. Amer. Math. Soc. **16**, 139–183 (2003)
[9] Kim, H., Sarnak, P.: Refined estimates towards the Ramanujan and Selberg conjectures. Appendix to Kim [8]
[10] Landau, E.: Zwei neue Herleitunggen für die asymptitische Anzahl der Primzahlen unter einer gegebenen Grenze. Sitz. Ber. Preuss. Akad. Wiss. Berlin, 1908, pp. 746–764
[11] Landau, E.: Über die Anzahl der Gitterpunkte in gewisser Bereichen. (Zweite Abhandlung) Gött. Nach., 1915, pp. 209–243

[12] Liu, J., Ye, Y.: Superposition of zeros of distinct L-functions. Forum Math. **14**, 419–455 (2002)
[13] Liu, J., Ye, Y.: Weighted Selberg orthogonality and uniqueness of factorization of automorphic L-functions. Forum Math. **17**, 493–512 (2005)
[14] Liu, J., Ye, Y.: Selberg's orthogonality conjecture for automorphic L-functions. Amer. J. Math. **127**, 837–849 (2005)
[15] Moeglin, C., Waldspurger, J.-L.: Le spectre résiduel de $GL(n)$. Ann. Sci. École Norm. Sup., (4) **22**, 605–674 (1989)
[16] Moreno, C.J.: Explicit formulas in the theory of automorphic forms. Lecture Notes Math. vol. **626**, Springer, Berlin, 1977, pp. 73–216
[17] Moreno, C.J.: Analytic proof of the strong multiplicity one theorem. Amer. J. Math. **107**, 163–206 (1985)
[18] Ram Murty, M.: Selberg's conjectures and Artin L-functions. Bull. Amer. Math. Soc. **31**, 1–14 (1994)
[19] Ram Murty, M.: Selberg's conjectures and Artin L-functions II. Current trends in mathematics and physics, Narosa, New Delhi, 1995, pp. 154–168
[20] Rudnick, Z., Sarnak, P.: Zeros of principal L-functions and random matrix theory. Duke Math. J. **81**, 269–322 (1996)
[21] Sarnak, P.: Non-vanishing of L-functions on $R(s) = 1$. In: Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, 2004, pp. 719–732
[22] Selberg, A.: Old and new conjectures and results about a class of Dirichlet series. Collected Papers, vol. II, Springer, 1991, pp. 47–63
[23] Shahidi, F.: On certain L-functions. Amer. J. Math. **103**, 297–355 (1981)
[24] Shahidi, F.: Fourier transforms of intertwining operators and Plancherel measures for $GL(n)$. Amer. J. Math. **106**, 67–111 (1984)
[25] Shahidi, F.: Local coefficients as Artin factors for real groups. Duke Math. J. **52**, 973–1007 (1985)
[26] Shahidi, F.: A proof of Langlands' conjecture on Plancherel measures; Complementary series for p-adic groups. Ann. Math. **132**, 273–330 (1990)