On the logarithmic zero-free regions of zeta and L-function.

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Abstract

For detecting the logarithmic zero-free regions of general classes of L-functions which satisfy a certain functional equation and have an Euler product. One always first construct a positive Dirichlet series $D(s)$ attached with this class of L-function by their Euler product. In their independent paper and book, Sarnak, Iwaniec and Kowalski showed that the zero-free region then followed by applying a lemma of Goldfeld, Hoffstein and Lieman to such $D(s)$ which is required to only have poles at $s = 1$. But for many applications, the positive Dirichlet series $D(s)$ will have other poles besides $s = 1$ on the line $\operatorname{Re} s = 1$. In this paper, we handle this case and establish lemmas for detecting logarithmic zero-free regions of very general classes of L-functions. In particular, we apply our results to Hecke L-functions for number fields and Rankin-Selberg L-function for classical cusp forms.

Keywords: L-function, Hecke zeta-function, zero free region, cusp forms

1 Introduction

In 1899, de la Vallée Poussin proved that the Riemann zeta function has a logarithmic zero-free region of the form

$$1 - \frac{c}{\log(2 + |t|)} < \sigma, \ t > 1,$$

[0]2000 Mathematics Subject Classification. Primary 11R42, 11F66.
[0]The first author is supported in part by China NSF Grant.

where $s = \sigma + it$ is a complex number and c is an absolute positive constant. The essence of his method was to compare the Hadamard product from the theory of entire functions with the defining, positive Dirichlet series derived from the Euler products. By the same method, the zero-free regions of the Dirichlet L-functions $L(s, \chi)$ can also be detected. We summarize the classical results for zero-free regions of $\zeta(s)$ and $L(s, \chi)$ (see [1] or [2]) as follows:

Theorem 1 *Let $s = \sigma + it$ and χ be a Dirichlet character modulo q. There exists a constant $c > 0$ such that the following is true:*

1. If χ is the principal character χ_0, then the zeroes of $L(s, \chi_0)$ in the critical strip $\operatorname{Re} s \in (0, 1]$ are equal to the zeroes of $\zeta(s)$, and $\zeta(s)$ has no complex zeroes in the region
$$1 - \frac{c}{\log(2 + |t|)} < \sigma. \tag{1}$$
Further, $\zeta(s)$ has no real zeroes in the interval $(0, 1)$ as shown by elementary methods in [3, p. 30].

2. If χ is a complex character, $L(s, \chi)$ has no zeroes in the region
$$\sigma \geq 1 - \frac{c}{\log q(|t| + 2)}. \tag{2}$$

3. If χ is a nonprincipal real character, $L(s, \chi)$ has no complex zeroes in the region
$$\sigma \geq 1 - \frac{c}{\log q(|t| + 2)}. \tag{3}$$
$L(s, \chi)$ has at most one simple real zero in this region. Such real zero is usually called the Siegel zero or exceptional zero.

4. If χ_1 and χ_2 are two distinct primitive characters modulo q_1, q_2 respectively, and β_1, β_2 are the real zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$, then
$$\min(\beta_1, \beta_2) < 1 - \frac{c}{\log q_1 q_2}.$$
The statement 4 implies the following:

   (a) There is at most one character χ modulo q such that the corresponding function $L(s, \chi)$ has a real zero β satisfying
$$1 > \beta \geq 1 - \frac{c}{\log q}.$$

   (b) For $2 \leq q \leq x$, there exists at most one q_1 with $2 \leq q_1 \leq x$ and at most one real primitive character χ_1 modulo q_1 such that $L(s, \chi_1)$ has a real simple zero β_1 satisfying
$$1 > \beta_1 \geq 1 - \frac{c}{\log x}.$$
Moreover, if χ is a real character modulo q and $L(s, \chi)$ has a real zero in the above range, then $q \equiv 0 \pmod{q_1}$.

More than a hundred years after their first proof, these results are essentially still the best known results for Dirichlet series $L(s,\chi)$, which were later understood as the L-functions of $GL(1)$. In 1994, Goldfeld, Hoffstein and Lieman [4] proved a remarkable fundamental lemma for $GL(n)$, $n \geq 2$, which allows to prove that many classes of automorphic L-functions do not have Siegel zeroes (see also [5] [6]). Recently, Sarnak [7], Iwaniec and Kowalski [8] showed that the logarithmic zero-free region for many classes of L-functions can also be obtained immediately by applying this lemma to a positive Dirichlet series $D(s)$ constructed from the class of the detected L-functions by Euler product. Even though this idea do not allow one to improve classical logarithmic zero-free regions, it provides a simple and elegant approach to establish zero-free regions of these or more complicated L-functions. The lemma of Goldfeld, Hoffstein and Lieman [4] requires that the constructed positive Dirichlet series only have poles at $s = 1$. For many applications, the positive Dirichlet series $D(s)$ will have other poles besides $s = 1$ on the line $\operatorname{Re} s = 1$. In this paper, we expand the theory of zero-free regions for L-functions by establishing lemmas for such positive Dirichlet series that have other poles besides $s = 1$. In particular, as applications, we establish the logarithmic zero-free regions for two classes of L-functions:

1. Dedekind zeta-functions and Hecke L-functions for arbitrary number fields (Theorem 2).
2. Rankin-Selberg L-functions attached to classical cusp forms (Theorem 3).

2 Fundamental Lemmas

We state the two fundamental Lemmas for detecting zero-free regions of L-functions. Lemma 1 (4) was first shown by Goldfeld, Hoffstein and Lieman [4, Lemma 1].

Lemma 1 *Let $\phi(s)$ be a Dirichlet series with nonnegative coefficients which is absolutely convergent for $\operatorname{Re} s > 1$. Suppose that $\phi(s)$ has an Euler product such that $\phi(s) \neq 0$ for $\operatorname{Re} s > 1$, and $\phi'(s)/\phi(s)$ is negative for real $s > 1$. Let $\phi(s)$ have a pole of order m at $s = 1$, and let $\Lambda(s) = s^m (1-s)^m G(s)\phi(s)$ satisfy $\Lambda(s) = \lambda\Lambda(1-s)$, where $\Lambda(s)$ is an entire function of order 1. Here,*

$$G(s) = D^s \prod_{i=1}^{r_1} \Gamma\left(\frac{s+c_i}{2}\right) \prod_{j=1}^{r_2} \Gamma(s+c_j)$$

where D is called the conductor of $\phi(s)$, λ is a complex constant with $|\lambda| = 1$, and the c_i's are complex constants. Suppose that the zeroes of $\Lambda(s)$ only occur in conjugate pairs in the critical strip, i.e. ρ and $\bar{\rho}$ are both zeroes of $\Lambda(s)$ and $\operatorname{Re}\rho \in [0,1]$. These zeroes are identical with the zeroes of $\phi(s)$ in the strip $\operatorname{Re}\rho \in (\frac{1}{2}, 1]$. Let $M = max(D,1)(2+\max\{|c_i|\}) \geq 2$, then there exists an effective constant c, depending only on r_1, r_2, and m, such that the following holds:

1. $\phi(s)$ has at most m real zeroes in the region

$$1-\frac{c}{\log M}<s. \tag{4}$$

2. $\phi(s)$ has at most $[\frac{5m}{8}]$ pairs of conjugate complex zeroes in the region

$$1-\frac{c}{\log M}<\operatorname{Re} s,\ \ |\operatorname{Im} s|<\frac{c^{\frac{1}{2}}}{2\log M}. \tag{5}$$

Proof. (4) can be seen from [4, Lemma 1], where the definition $M=1+D\max\{|c_l|\}$ should be a misprint since $c_l=0$ will cause infinity at the left-hand side of (4). The assertion (5) follows from Lemma 2. ■

In lemma 1, $\phi(s)$ is supposed to be a Dirichlet series with nonnegative coefficients, and $\phi'(s)/\phi(s)<0$ for real $s>1$. In many applications, the second condition, i.e., $\phi'(s)/\phi(s)<0$ for real $s>1$ will implythe first condition that $\phi(s)$ is a Dirichlet series with nonnegative coefficients. We call such Dirichlet series $\phi(s)$ the *positive* Dirichlet series. Such positive Dirichlet series appears essentially in the method of de la Vallée Poussin which now can be understood as: we always first construct a positive Dirichlet series $D(s)$ from the class of the detected L-functions, and then apply Lemma 1 or 2 to $D(s)$ to obtain the logarithmic zero-free region immediately.

Sarnak [7], Iwaniec and Kowalski [8, p. 105] used this idea and Lemma 1 of Goldfeld, Hoffstein and Lieman [4] to obtain the logarithmic zero-free region for many classes of L-function. But for many applications, the constructed positive Dirichletseries $D(s)$ will have poles not only at $s=1$, but also possibly at $s=1\pm i\tau_j$, $\tau_j\in\mathbb{R}^+$. For example, see the classical case of Dirichlet L-function $L(s,\chi)$ while χ is a real character (see (19)). Hence Lemma 1 will not be sufficient in these cases. In this paper, we establish the following lemma for handling the positive Dirichlet series which has other poles besides $s=1$ on the line $\operatorname{Re} s=1$.

Lemma 2 *Let $\phi(w)$ be a Dirichlet series which is absolutely convergent for $\operatorname{Re} w>1$, and $\phi(w)$ has an Euler product such that $\phi(w)\neq 0$ for $\operatorname{Re} w>1$. Furthermore, suppose $\phi(w)$ also satisfies the following:*

1. *Positivity condition: $\phi(w)$ is a positive Dirichlet series, i.e., $\phi(w)$ is a Dirichlet series with nonnegative coefficients, and $\phi'(w)/\phi(w)$ is negative for real $w>1$.*

2. *Functional Equation: Let $\phi(w)$ have a pole of order m at $w=1$, and other poles counted with multiplicity at $w=1\pm i\tau_j,\tau_j\in\mathbb{R}^+$, $j=1,\ldots,\kappa$, where $\kappa=0$ means that no such pole exists. Let*

$$\Lambda(w)=w^m(1-w)^m\prod_{j=1}^{\kappa}(w+i\tau_j)(w-i\tau_j)(1+i\tau_j-w)(1-i\tau_j-w)$$
$$\times G(w)\phi(w)$$

satisfy the functional equation

$$\Lambda(w) = \lambda \Lambda(1-w), \tag{6}$$

where $\Lambda(w)$ *is an entire function of order* 1, *and* λ *is a complex constant with* $|\lambda| = 1$. *Here,*

$$G(w) := D^w \prod_{j=1}^{r_0} \Gamma\left(\frac{w + c_{0,j}}{d_{0,j}}\right) \prod_{j=1}^{\kappa} \prod_{i=1}^{r_j} \Gamma\left(\frac{w + i\tau_j + c_{i,j}}{d_{i,j}}\right) \Gamma\left(\frac{w - i\tau_j + c_{i,j}}{d_{i,j}}\right),$$

where D *is called the conductor of* $\phi(w)$, $d_{i,j} = 1, 2$ *and* $c_{i,j}$ *are complex constants. Suppose that the zeroes of* $\Lambda(w)$ *occur only in conjugate pairs in the critical strip, i.e.* ρ *and* $\bar{\rho}$ *are both zeroes of* $\Lambda(w)$, *and* $\operatorname{Re}\rho \in [0,1]$. *We further assume that these zeroes are identical with the zeroes of* $\phi(w)$ *in the strip* $(\frac{1}{2}, 1]$.

Let

$$M_0 = max(D,1) \prod_{i,j} (2 + |c_{i,j}|) \geq 2,$$
$$M = max(D,1) \prod_{j} \prod_{i} (2 + |c_{i,j}| + |\tau_j|) \geq 2,$$

which are called "the analytic conductors" of the corresponding L-function $\phi(w)$. *Then, there exists an effective constant* c *depending only on* m, κ *such that the following is true:*

1. *If* $\kappa = 0$ *or* $\tau_j \geq 2\kappa c^{\frac{1}{4}} / \log M_0$ *for* $\kappa \geq 1$, $\phi(w)$ *has at most* m *real zeroes in the region*
$$1 - \frac{c}{\log M} < w \leq 1. \tag{7}$$

2. *If* $\kappa = 0$ *or* $\tau_j < 2$ *for* $\kappa \geq 1$, $\phi(w)$ *has at most* $[\frac{5(m+2\kappa)}{8}]$ *pairs of conjugate complex zeroes in the region*
$$1 - \frac{c}{\log M_0} < \operatorname{Re} w \leq 1, \quad |\operatorname{Im} w| \leq \frac{c^{\frac{1}{2}}}{2 \log M_0}. \tag{8}$$

Proof. $\Lambda(w)$ is an entire function of order 1. Using the Hadamard product, we can write

$$\Lambda(w) = e^{A+Bw} \prod_{\rho} \left(1 - \frac{w}{\rho}\right) e^{w/\rho},$$

where ρ runs over the zeroes of $\Lambda(w)$. Taking the logarithmic derivative, we obtain

$$\frac{\Lambda'(w)}{\Lambda(w)} = B + \sum_{\rho} \frac{1}{w - \rho} + \sum_{\rho} \frac{1}{\rho}. \tag{9}$$

From the functional equation,

$$\frac{\Lambda'(w)}{\Lambda(w)} = -\frac{\Lambda'(1-w)}{\Lambda(1-w)},$$

we see that

$$2B + 2\sum_{\rho}\frac{1}{\rho} = \sum_{\rho}\frac{1}{w-(1-\rho)} - \sum_{\rho}\frac{1}{w-\rho} = 0.$$

since ρ and $1-\rho$ are both zeroes of $\Lambda(w)$ by the functional equation (6). Hence, we get from (9) that for real $w > 1$,

$$\begin{aligned}\sum_{\rho}\frac{1}{w-\rho} &= \frac{\Lambda'(w)}{\Lambda(w)} \qquad (10)\\ &= \frac{m}{w-1} + 2\sum_{j=1}^{\kappa}\frac{w-1}{(w-1)^2+\tau_j^2}\\ &+ \sum_{j=1}^{r_0}\frac{1}{d_{0,j}}\frac{\Gamma'}{\Gamma}\left(\frac{w+c_{0,j}}{d_{0,j}}\right) + \sum_{j=1}^{\kappa}\sum_{i=1}^{r_j}\frac{1}{d_{i,j}}\frac{\Gamma'}{\Gamma}\left(\frac{w+i\tau_j+c_{i,j}}{d_{i,j}}\right)\\ &+ \sum_{j=1}^{\kappa}\sum_{i=1}^{r_j}\frac{1}{d_{i,j}}\frac{\Gamma'}{\Gamma}\left(\frac{w-i\tau_j+c_{i,j}}{d_{i,j}}\right) + \frac{\phi'(w)}{\phi(w)} + A_0\log D_1,\end{aligned}$$

where A_0 is an absolute constant depending on m, κ and $D_1 = \max(D, 1)$. For real $w > 1$, since $\rho = \beta + i\gamma$ and $\bar{\rho} = \beta - i\gamma$ are both zeroes of $\Lambda(w)$ and lie in the critical strip $[0, 1]$, we see

$$\frac{1}{w-\rho} + \frac{1}{w-\bar{\rho}} = 2\frac{w-\beta}{(w-\beta)^2+\gamma^2} > 0.$$

Thus, if we denote by β_i the real zeroes of $\phi(w)$ in the critical strip, we have for real $w > 1$,

$$\begin{aligned}\sum_{i=1}^{n}\frac{1}{w-\beta_i} &\le \frac{m}{w-1} + 2\sum_{j=1}^{\kappa}\frac{w-1}{(w-1)^2+\tau_j^2} + A_0\log D_1 \qquad (11)\\ &+ \sum_{j=1}^{r_0}\left|\frac{1}{d_{0,j}}\frac{\Gamma'}{\Gamma}\left(\frac{w+c_{0,j}}{d_{0,j}}\right)\right| + \sum_{j=1}^{\kappa}\sum_{i=1}^{r_j}\left|\frac{1}{d_{i,j}}\frac{\Gamma'}{\Gamma}\left(\frac{w+c_{i,j}\pm i\tau_j}{d_{i,j}}\right)\right|,\end{aligned}$$

by the assumption $\frac{\phi'(w)}{\phi(w)} < 0$. We know from (15) in [9] p. 53 that

$$\left|\frac{\Gamma'(w)}{\Gamma(w)} - \log w - \frac{1}{2w}\right| \le \frac{1}{6|w|^2},$$

which implies

$$\left|\frac{\Gamma'(w)}{\Gamma(w)}\right| \leq \log|w| + O\left(\frac{1}{|w|}\right) .$$

Thus, we see from (11) that there exists a constant A depending on m, κ such that for $w \in (1,2)$,

$$\sum_{i=1}^{n} \frac{1}{w-\beta_i} \leq \frac{m}{w-1} + 2\sum_{j=1}^{\kappa} \frac{w-1}{(w-1)^2+\tau_j^2} + A\log M. \tag{12}$$

Let $w = 1+\delta/\log M$, and take $\delta < \min\{\frac{1}{4}A^{-1}, 1\}$, $\tau_j \geq 2\kappa\delta^{\frac{1}{2}}/\log M_0$. We can assume that $\beta_i > 1 - c/\log M$ in (12) and the number of such β_i is n. Therefore,

$$\begin{aligned} n\frac{\log M}{\delta+c} &\leq \frac{m\log M}{\delta} + \frac{\log M}{\delta}\frac{2\kappa}{1+(4\kappa^2\log^2 M)/(\delta\log^2 M_0)} + \frac{\log M}{4\delta} \\ &\leq \frac{\log M}{\delta}\left(m + \frac{1}{4} + \frac{2\kappa}{1+(4\kappa^2\log^2 M)/(\delta\log^2 M_0)}\right). \end{aligned}$$

Hence,

$$n \leq \left(m + \frac{1}{4} + \frac{2\kappa}{1+(4\kappa^2\log^2 M)/(\delta\log^2 M_0)}\right)\left(1+\frac{c}{\delta}\right).$$

Therefore, if c is chosen small enough and $\delta = c^{\frac{1}{2}}$, a contradiction is obtained whenever $n \geq m+1$. This proves (7) and also Lemma 1 (4).

For the proof of (8), we denote by n_1 the number of pairs of conjugate zeroes $\rho_i = \beta_i \pm i\gamma_i$ and by n_2 the number of real zeroes β'_j. By the same argument as before, we have for real $w > 1$, $\tau_j < 2$,

$$\begin{aligned} \sum_{i=1}^{n_1} \frac{2(w-\beta_i)}{(w-\beta_i)^2+\gamma_i^2} + \sum_{j=1}^{n_2} \frac{1}{w-\beta'_j} &\leq \frac{m}{w-1} + 2\sum_{j=1}^{\kappa} \frac{w-1}{(w-1)^2+\tau_j^2} + A\log M_0 \\ &\leq \frac{m+2\kappa}{w-1} + A\log M_0. \end{aligned}$$

If $|\gamma_i| \leq \frac{\delta}{2\log M_0} \leq \frac{1}{2}(w-1) \leq \frac{1}{2}(w-\beta_i)$, we have

$$\sum_{i=1}^{n_1} \frac{2}{(w-\beta_i)(1+1/4)} + \sum_{j=1}^{n_2} \frac{1}{w-\beta'_j} \leq \frac{m+2\kappa}{w-1} + A\log M_0.$$

We suppose $\beta_i, \beta'_j > 1 - c/\log M_0$, and take $w = 1+\delta/\log M_0$ with $\delta < \min(\frac{1}{4}A^{-1}, 1)$. Hence,

$$\frac{1}{(c+\delta)}\left(\frac{8n_1}{5} + n_2\right)\log M_0 \leq \frac{m+2\kappa+\frac{1}{4}}{\delta}\log M_0,$$

which implies

$$\frac{8n_1}{5} + n_2 \leq \left(m + 2\kappa + \frac{1}{4}\right)\left(1+\frac{c}{\delta}\right). \tag{13}$$

In particular, we get $n_1 \leq \frac{5(m+2\kappa)}{8}$, when c is chosen sufficiently small and $\delta = c^{\frac{1}{2}}$. ■

3 Application to Dedekind zeta-function and Hecke L-function

Let K be an algebraic number field of degree $[K:\mathbb{Q}]=n=r_1+2r_2$ (in the standard notation), d the absolute value of the discriminant, h its class number, and $r=r_1+r_2-1$ its number of fundamental units. Let $\mathcal{E}$, R, and ω denote respectively the group of units, the regulator, and the number of roots of unity in K.

Application to Dedekind zeta-function. We denote the Dedekind zeta-function as

$$\zeta_K(s)=\sum_{\mathfrak{a}\in O_K}\frac{1}{N\mathfrak{a}^s}=\prod_{\mathfrak{p}}\left(1-\frac{1}{N\mathfrak{p}^s}\right)^{-1},$$

which is absolutely convergent for $\operatorname{Re} s>1$. $\zeta_K(s)$ has an analytic continuation and functional equation [10, p. 259], i.e., $\zeta_K(s)$ is analytic everywhere except a simple pole at $s=1$ with residue

$$\operatorname{Res}=\frac{2^{r_1}(2\pi)^{r_2}}{\omega d^{1/2}}hR.$$

We set

$$\Lambda_K(s)=D_K^s\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\zeta_K(s)$$

with

$$D_K=2^{-r_2}d^{1/2}\pi^{-n/2}.$$

The functional equation is

$$\Lambda_K(s)=\Lambda_K(1-s), \tag{14}$$

and $\Lambda_K(s)$ has a simple pole at $s=0$ and $s=1$ with polar part

$$\frac{2^{r_1}hR}{\omega}\left(\frac{1}{s-1}-\frac{1}{s}\right)=\frac{2^{r_1}hR}{\omega}\frac{1}{s(s-1)}.$$

We set

$$\phi(w)=\zeta_K^6(w)\zeta_K^4(w+it)\zeta_K^4(w-it)\zeta_K(w+2it)\zeta_K(w-2it).$$

In order to apply Lemma 2, we prove that $\phi(w)$ is a positive Dirichlet series. Since it is absolutely convergent for $\operatorname{Re} w>1$, we see from the Euler product

$$-\frac{\phi'(w)}{\phi(w)}=\sum_{\mathfrak{a}\in O_K}\Lambda_K(\mathfrak{a})(N\mathfrak{a})^{-w}\left(6+8\cos(t\ln(N\mathfrak{a}))+2\cos(2t\ln(N\mathfrak{a}))\right),$$

$$\Lambda_K(\mathfrak{a})=\begin{cases}\log N\mathfrak{p} & \text{if } \mathfrak{a}=\mathfrak{p}^l,\ l\geq 1,\\ 0 & \text{otherwise.}\end{cases}$$

Hence, $\phi(w)$ satisfies the positivity condition of Lemma 2 by the trigonometric inequality

$$3+4\cos\theta+\cos 2\theta\geq 0.$$

Set

$$Q_K = \max\left(2^{-r_2} d^{1/2} \pi^{-n/2}, 1\right). \tag{15}$$

By the functional equation of $\zeta_K(s)$ (14), it is easily seen that $\kappa = 5$ and the analytic conductor is $M = Q_K(|t|+2)^{r_1+r_2}$, $M_0 = Q_K$ with the notation κ, M and M_0 as in Lemma 2. We now suppose that for $s = w + it, \operatorname{Re} s = w$ there is $\zeta_K(s) = \zeta_K(w+it) = 0$, with

$$1 - \frac{c_1}{\log M} < w \leq 1, \ \ |t| > \frac{10 c_1^{\frac{1}{4}}}{\log M_0}. \tag{16}$$

$\phi(w)$ has at least 8 real zeroes (counted with multiplicity) in the region (16), but only $m = 6$, i.e. $\phi(w)$ has a pole of order 6 at $s = 1$, which contradicts Lemma 2 (7). Thus, $\zeta_K(s)$ has no zeroes in the region (16). In order to consider the complex zeroes with an imaginary part of small absolute value, we suppose that $\zeta_K(w) = 0$ in the region

$$1 - \frac{c_2}{\log M_0} < \operatorname{Re} w \leq 1, \ 0 < |\operatorname{Im} w| \leq \frac{c_2^{\frac{1}{2}}}{2 \log M_0}.$$

Taking $\phi(w) = \zeta_K(w)$, we see that $\phi(w)$ has at least 1 pair of conjugate complex zeroes in the above region, but only 1 pole at $w = 1$ and $\kappa = 0$ which is a contradiction to Lemma 2 (8). By choosing $10 c_1^{\frac{1}{4}} \leq c_2^{\frac{1}{2}}/2$, we prove that $\zeta_K(s)$ has no complex zeroes in the region

$$1 - \frac{c}{\log Q_K (|t|+2)^{r_1+r_2}} < \operatorname{Re} s \leq 1,$$

where c is an absolute positive constant. If the real zeroes exists in the above region, it must be single and simple, otherwise $\zeta_K(s)$ has at least two zeroes but only one pole which contradicts Lemma 1. ■

Applications to Hecke L-functions. Research on Hecke L-functions has followed two main approaches: The first is Hecke's multidimensional method by using Grössencharacters and prime ideal number (see [11] or [12, p. 493]). The second approach is based on Tate's thesis [13] and uses Adelic analysis. Whereas Coleman [14] has proved a Vinogradov-type result of a zero-free region for Hecke L-function by using the first approach, we will follow Tate's approach and in particular we will make use of the formulation of the functional equation as given by Andre Weil [15].

Let χ be a Hecke character (unitary) of $K_A^\times / K^\times$ with conductor $\mathfrak{f} = \prod \mathfrak{p}_v^{f(v)}$, where $f(v) = 0$ if and only if χ_v is unramified, i.e., $\mathfrak{p}_v \nmid \mathfrak{f}$. We denote the Hecke L-functions by

$$L_K(s, \chi) = \prod_{\mathfrak{p}_v \nmid \mathfrak{f}} \left(1 - \chi(\pi_v) N(\mathfrak{p}_v)^{-s}\right)^{-1} \ \text{ for } \operatorname{Re} s > 1.$$

where π_v is the prime element of the finite places K_v, and note that $\chi(\pi_v)$ is independent of the choice of the prime element π_v for unramified places v

satisfying with $\mathfrak{p}_v \nmid \mathfrak{f}$. By [15, p. 116, Corollary 2], $\chi(a) = \chi'(a)|a|_A^{i\tau}$, where χ' is a Hecke character of finite order, and then $L_K(s,\chi) = L_K(s+i\tau,\chi')$. Further, if $\chi(a) = |a|_A^{-i\tau_0}$, then $L_K(s,\chi) = \zeta_K(s-i\tau_0)$ has no zeroes in the region

$$1 - \frac{c}{\log Q_K(|t-\tau_0|+2)} < \operatorname{Re} s \leq 1,$$

except a single and simple zero at the line $t = \tau_0$. In the following, without loss of generality, we will always assume that χ is a Hecke character with finite order. Hence, $L_K(s,\chi)$ has an analytic continuation and is analytic everywhere unless χ is principal, in which case $L_K(s,\chi) = \zeta_K(s)$ has a simple pole at $s = 1$. We write

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right),$$
$$\Gamma_{\mathbb{C}}(s) = (2\pi)^{1-s}\Gamma(s),$$

and

$$\Lambda_K(s,\chi) = D^{-s}\prod_{v=1}^{r_1}\Gamma_{\mathbb{R}}(s+\lambda_v)\prod_{v=r_1+1}^{r_1+r_2}\Gamma_{\mathbb{C}}(s+\lambda_v)L_K(s,\chi),$$

where the infinite parameter λ_v is defined in [15, p. 130] as

$$\begin{aligned}
&\chi_v(x) = x^{-N_v}|x|^{\lambda_v},\ N_v = 0,1, && \text{if } v \text{ is real places, } v = 1,\ldots,r_1;\\
&\chi_v(x) = \begin{cases} x^{-N_v}|x|^{2\lambda_v} \\ \text{or } \bar{x}^{-N_v}|x|^{2\lambda_v} \end{cases} \quad 0 \leq N_v \in \mathbb{Z}, && \text{if } v \text{ is imaginary places, } v = r_1+1,\ldots,r_1+r_2,
\end{aligned} \tag{17}$$

and

$$D = \sqrt{dN(\mathfrak{f})},$$
$$N(\mathfrak{f}) = \prod_{\mathfrak{p}_v|\mathfrak{f}} N(\mathfrak{p}_v)^{f_v}.$$

The functional equation is given by

$$\Lambda_K(s,\chi) = \lambda\Lambda_K\left(1-s,\chi^{-1}\right), \tag{18}$$

where λ is a complex constant depending on χ with $|\lambda| = 1$. We set

$$\phi(w) = \zeta_K^6(w)L_K^4(w+it,\chi)L_K^4(w-it,\bar{\chi})L_K\left(w+2it,\chi^2\right)L_K\left(w-2it,\bar{\chi}^2\right), \tag{19}$$

which implies

$$\begin{aligned}
\frac{\phi'(w)}{\phi(w)} &= 3\frac{\zeta_K'(w)}{\zeta_K(w)} + 4\frac{L'(w+it,\chi)}{L(w+it,\chi)} + \frac{L'\left(w+2it,\chi^2\right)}{L(w+2it,\chi^2)}\\
&\quad + 3\frac{\zeta_K'(w)}{\zeta_K(w)} + 4\frac{L'(w-it,\bar{\chi})}{L(w-it,\bar{\chi})} + \frac{L'\left(w-2it,\bar{\chi}^2\right)}{L(w-2it,\bar{\chi}^2)}.
\end{aligned}$$

Using

$$-\frac{L_K'(w,\chi)}{L_K(w,\chi)} = \sum_{\substack{\mathfrak{a} e O_K \\ (a,\mathfrak{f})=1}} \frac{\chi(\mathfrak{a})\Lambda_K(\mathfrak{a})}{|N\mathfrak{a}|^w},$$

which is absolutely convergent for $\operatorname{Re} w > 1$, we obtain

$$\begin{aligned}-\frac{\phi'(w)}{\phi(w)} &= 2\sum_{\substack{\mathfrak{a} e O_K \\ (\mathfrak{a},\mathfrak{f})=1}} \Lambda_K(\mathfrak{a})|N\mathfrak{a}|^{-w} \times \\ &\times \operatorname{Re}\{3+4\cos(t\log|N\mathfrak{a}|+\arg\chi(\mathfrak{a}))+\cos(2(t\log|N\mathfrak{a}|+\arg\chi(\mathfrak{a})))\} \\ &+6\sum_{\substack{\mathfrak{a} e O_K \\ (\mathfrak{a},\mathfrak{f})\neq 1}} \Lambda_K(\mathfrak{a})|N\mathfrak{a}|^{-w},\end{aligned}$$

where all coefficients are nonnegative. Hence, $\phi(w)$ also satisfies the positivity condition of our fundamental Lemmas. In view of the functional equation (18) and the definitions in Lemma 2, we define the analytic conductors as:

$$M = Q_K N(\mathfrak{f}) \prod_{v=1}^{r_1+r_2} (|t|+|\lambda_v|+2),\ M_0 = Q_K N(\mathfrak{f}) \prod_{v=1}^{r_1+r_2} (|\lambda_v|+2),$$

where Q_K is defined as in (15). When $\chi^2 \neq \chi_0$, i.e. $\chi^2 \neq 1$, (since by the weak approximation theorem, if the Hecke character χ_0 is trivial for almost all finite places, it must be trivial for all places), we call such χ the complex characters. We assume that $L(s) = L(w+it,\chi) = 0$, $\operatorname{Re} s = w$ where

$$1-\frac{c}{\log M} < w \leq 1,\ t \in \mathbb{R}.$$

Hence, $\phi(w)$ has at least 8 zeroes (counted with multiplicity), but only 6 poles at $s=1$ and $\kappa=0$, which contradicts Lemma 1 (4). Thus for any complex Hecke character of finite order χ, there exists a constant $c>0$ such that

$$L_K(s,\chi) \neq 0 \text{ for } 1-\frac{c}{\log M} < \operatorname{Re} s \leq 1.$$

For non-principal real characters χ satisfying $\chi^2 = \chi_0$, $\phi(w)$ has 6 poles at $s=1$ and $\kappa=1$. Hence, by Lemma 2, (7) $L(s,\chi)$ has no complex zeroes in the region

$$1-\frac{c_1}{\log M} < \operatorname{Re} s,\ \ |\operatorname{Im} s| > \frac{2c_1^{\frac{1}{4}}}{\log M_0},$$

as otherwise it would have 8 zeroes, but $m=6$ which is a contradiction. Similarly, there are no complex zeroes in the region

$$1-\frac{c_2}{\log M_0} < \operatorname{Re} s,\ \ |\operatorname{Im} s| \leq \frac{c_2^{\frac{1}{2}}}{2\log M_0},$$

because otherwise, when taking $t=0$, $\phi(w)$ has at least 8 pairs of complex zeroes but at most 8 poles at $w=1$, which contradicts Lemma 2 (8). Choosing $2c_1^{\frac{1}{4}} \leq c_2^{\frac{1}{2}}/2$, we prove that there exists an absolute constant $c>0$ such that for any real Hecke character of finite order χ, i.e, $\chi^2=\chi_0$, $L_K(s,\chi)$ has no complex zeroes in the region

$$1-\frac{c}{\log M}<\operatorname{Re} s\leq 1.$$

There exists at most one real, simple zero β of $L_K(s,\chi)$ in this region. Otherwise, taking $t=0$, $\phi(w)$ must have at least 16 zeroes, but only 8 poles which is a contradiction to Lemma 1.

We now consider two distinct χ_1 and χ_2 Hecke characters, which are both of finite order and with conductor $\mathfrak{f}_1,\mathfrak{f}_2$ respectively, and where β_1 and β_2 are real zeroes of $L(s,\chi_1), L(s,\chi_2)$ in the region

$$1-\frac{c}{\log\left(Q_K N(\mathfrak{f}_1)N(\mathfrak{f}_2)\prod_{v=1}^{r_1+r_2}(|\lambda_v|+2)\right)}<s. \tag{20}$$

We set

$$\begin{aligned}\phi(s)=\zeta_K^2(s)\,L_K(s,\chi_1)\,L_K(s,\chi_2)\,L_K(s,\chi_1\chi_2)\\ \times L_K(s,\bar{\chi}_1)\,L_K(s,\bar{\chi}_2)\,L_K(s,\bar{\chi}_1\bar{\chi}_2).\end{aligned} \tag{21}$$

Thus,

$$\begin{aligned}-\frac{\phi'(s)}{\phi(s)}=2\operatorname{Re}\sum_{\substack{\mathfrak{a}\in O_K\\(\mathfrak{a},\mathfrak{f})=1}}\Lambda_K(\mathfrak{a})(1+\chi_1(\mathfrak{a}))(1+\chi_2(\mathfrak{a}))|N\mathfrak{a}|^{-s}\\ +2\sum_{\substack{\mathfrak{a}\in O_K\\(\mathfrak{a},\mathfrak{f})\neq 1}}\Lambda_K(\mathfrak{a})|N\mathfrak{a}|^{-s}.\end{aligned}$$

Thus, $\phi(s)$ is a positive Dirichlet series and satisfies the desired functional equation from (18). By our assumption, $\phi(s)$ has at least 4 zeroes in the region (20), but only 2 poles which is a contradiction to Lemma 1. Hence, we must have

$$\min(\beta_1,\beta_2)<1-\frac{c}{\log\left(Q_K N(\mathfrak{f}_1)N(\mathfrak{f}_2)\left(\prod_{v=1}^{r_1+r_2}|\lambda_v|+2\right)\right)}.$$

■

We summarize our results concerning zero-free regions for Dedekind zeta-functions and Hecke L-functions as follows:

Theorem 2 *Let $s=\sigma+it$, χ be a unitary Hecke character of $K_A^\times/K^\times$ with conductor $\mathfrak{f}=\prod\mathfrak{p}_v^{f(v)}$, and $\chi(a)=\chi'(a)|a|_A^{-i\tau_0}$, $\tau_0\in\mathbb{R}$ where χ' a Hecke character of finite order. Then there exists an absolute constant $c>0$ such that the following is true:*

1. If χ is a complex character, i.e., $(\chi')^2 \neq 1$, then $L_K(s,\chi)$ has no zeroes in the region

$$\sigma \geq 1 - \frac{c}{\log\left(Q_K N(\mathfrak{f}) \prod_{v=1}^{r_1+r_2} (|t-\tau_0| + |\lambda_v| + 2)\right)}, \tag{22}$$

where λ_v is the infinite parameter attached with χ' in (17) and Q_K in (15).

2. If χ is a real character, i.e., $(\chi')^2 = 1$, then $L_K(s,\chi)$ has no zeroes in the region

$$\sigma \geq 1 - \frac{c}{\log\left(Q_K N(\mathfrak{f}) \prod_{v=1}^{r_1+r_2} (|t-\tau_0| + |\lambda_v| + 2)\right)}, \tag{23}$$

except a possible single, simple zero at the line $t = \operatorname{Im} s = \tau_0$.

3. Let χ_1 and χ_2 be two distinct Hecke characters of finite order and with conductors $\mathfrak{f}_1, \mathfrak{f}_2$ respectively. Let β_1 and β_2 be real zeroes of $L_K(s,\chi_1)$ and $L_K(s,\chi_2)$, then

$$\min(\beta_1,\beta_2) < 1 - \frac{c}{\log\left(Q_K N(\mathfrak{f}_1) N(\mathfrak{f}_2) \prod_{v=1}^{r_1+r_2} (|\lambda_v| + 2)\right)}.$$

Following the argument in [1, p. 130], the statement 3 implies the following:

(a) There is at most one character χ of finite order with conductor $\mathfrak{f}$ such that the corresponding function $L_K(s,\chi)$ has a real zero β satisfying

$$1 - \frac{c}{\log\left(Q_K N(\mathfrak{f}) \prod_{v=1}^{r_1+r_2} (|\lambda_v| + 2)\right)} < \beta < 1.$$

(b) For all Hecke characters χ of finite order that satisfy $2 \leq N(\mathfrak{f}_\chi) \leq x$ and the infinite parameter satified with $\prod_{v=1}^{r_1+r_2} (|\lambda_v| + 2) < Bx^A$, there exists at most one such character χ_1 that $L_K(s,\chi_1)$ has a real simple zero β_1 satisfying

$$1 - \frac{c}{\log(Q_K x)} < \beta_1 < 1,$$

where c depends on A, B.

Remark 1 *Theorem 1 follows directly from Theorem 2. Furthermore, using the analytic class number formula for Dedekind zeta-function of the cyclotomic field, the assertion of Theorem 1, 4a can be derived from the relation (23).*

Remark 2 *For Riemann zeta-function, it is easily to prove that $\zeta(s)$ has no real zeroes in the interval $(0,1)$ by elementary methods (see [3, p. 30]). But it is far beyond the temporary method to even prove that the Dedekind zeta-function $\zeta_K(s)$ has no Siegel zeroes, i.e., the real zero very near $s=1$. This is a very important open problem in analytic number theory, which will show that the Dirichlet L-function $L(s,\chi)$ has no Siegel zeroes by the analytic class number formula for Dedekind zeta-function of the cyclotomic field.*

4 Applications to Rankin-Selberg L-function

In this section, we establish zero-free regions for the Rankin-Selberg L-function of classical cusp forms. In the Remark 4, zero-free regions for Rankin-Selberg L-functions of irreducible cusp representations of $GL_m(\mathbb{A}_K)$, $GL_{m'}(\mathbb{A}_K)$ are discussed.

Let $S_k(\Gamma)$ be the space of cusp forms of weight k for $\Gamma := \Gamma(1) := SL_2(\mathbb{Z})$. (Note that k must be even.) Let $f \in S_k(\Gamma)$ and $g \in S_l(\Gamma)$ be normalized Hecke eigenforms defined as in [16, p. 299]. For the case of $\Gamma := SL_2(\mathbb{Z})$, the notion of "normalized" simply means that the first Fourier coefficient of $f(z)$ is $a_f(1) = 1$. We write the Fourier series expansion of the normalized Hecke eigenform $f \in S_k(\Gamma)$ at the cusp ∞ as

$$f(z) = \sum_{n=1}^{\infty} a_f(n)\, e(nz).$$

$a_f(n)$ must be real by the theory of Hecke operators. We define the corresponding L-functions for $\operatorname{Re} s > \frac{k+1}{2}$ as

$$\begin{aligned} L_f(s) &= \sum_{n=1}^{\infty} a_f(n)\, n^{-s} = \prod_p \left(1 - a_f(p)\, p^{-s} + p^{k-1-2s}\right)^{-1} \\ &= \prod_p \left(1 - \alpha_f(p)\, p^{-s}\right)^{-1} \left(1 - \bar{\alpha}_f(p)\, p^{-s}\right)^{-1}, \end{aligned}$$

where the last equality is the Peterson-Ramanujan conjecture which has been proved by P. Delign, and then obviously $|\alpha_f(p)| = p^{\frac{k-1}{2}}$. For a normalized Hecke eigenform $g \in S_l(\Gamma)$, we can define $L_g(s)$ similarly. The Rankin-Selberg L-function attached to f and g is defined as

$$\begin{aligned} L_{f\otimes g}(s) = &\prod_p \left(1 - \alpha_f(p)\, \alpha_g(p)\, p^{-s}\right)^{-1} \left(1 - \bar{\alpha}_f(p)\, \bar{\alpha}_g(p)\, p^{-s}\right)^{-1} \\ &\times \left(1 - \bar{\alpha}_f(p)\, \alpha_g(p)\, p^{-s}\right)^{-1} \left(1 - \alpha_f(p)\, \bar{\alpha}_g(p)\, p^{-s}\right)^{-1}, \end{aligned}$$

for $\operatorname{Re} s > \frac{k+l}{2}$. We also define the twisted Rankin-Selberg L-function for $\operatorname{Re} s >$

$\frac{k+l}{2}$ as

$$L_{f\otimes g}(s,\chi) := \prod_p \left(1-\alpha_f(p)\alpha_g(p)\chi(p)p^{-s}\right)^{-1}\left(1-\bar{\alpha}_f(p)\bar{\alpha}_g(p)\chi(p)p^{-s}\right)^{-1}$$
$$\times\left(1-\bar{\alpha}_f(p)\alpha_g(p)\chi(p)p^{-s}\right)^{-1}\left(1-\alpha_f(p)\bar{\alpha}_g(p)\chi(p)p^{-s}\right)^{-1} \quad ,$$

where χ is a Dirichlet character modulo q. Suppose that χ^* modulo q^* is the primitive character which induces χ, from the following relation

$$L_{f\otimes g}(s,\chi) = L_{f\otimes g}(s,\chi^*)\prod_{p|q}\left(1-\alpha_f(p)\alpha_g(p)\chi^*(p)p^{-s}\right)\left(1-\bar{\alpha}_f(p)\bar{\alpha}_g(p)\chi^*(p)p^{-s}\right)$$
$$\times\left(1-\bar{\alpha}_f(p)\alpha_g(p)\chi^*(p)p^{-s}\right)\left(1-\alpha_f(p)\bar{\alpha}_g(p)\chi^*(p)p^{-s}\right), \tag{24}$$

it is seen that we only need to consider the zero-free region of $L_{f\otimes g}(s,\chi^*)$. For the primitive character $\chi \bmod q$, Winnie Li [17, Theorem 2.2] has shown the following functional equation for the function $L_{f\otimes g}(s,\chi)$. Without loss of generality, we assume $k \geq l$ and write

$$\Lambda_{f\otimes g}(s,\chi) := \left(\frac{2\pi}{q}\right)^{-2s}\Gamma\left(s+\frac{k-l}{2}\right)\Gamma\left(s+\frac{k+l}{2}-1\right)L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi\right).$$

The functional equation can be written as

$$\Lambda_{f\otimes g}(s,\chi) = C_\chi \Lambda_{f\otimes g}(1-s,\bar{\chi}), \tag{25}$$

with the complex constant $|C_\chi| = 1$ and $\Lambda_{f\otimes g}(s,\chi)$ is an entire function except for simple poles at $s = 0, 1$ if and only if $f = g$ and χ is a principle character. From the functional equation and Euler Products, one easily derives that

1. $L_{f\otimes g}(s,\chi)$ has no zero for $\operatorname{Re} s > \frac{k+l}{2}$.
2. $L_{f\otimes g}(s,\chi)$ has a zero of order two at $s = -n$ for $n \in \mathbb{N}\cup\{0\}$, and has simple poles at the integers $s = 1,\ldots,l-1$.

The zeroes of $L_{f\otimes g}(s,\chi)$ in $\operatorname{Re} s \leq \frac{k+l}{2}-1$ are called the trivial zeroes, and $(\frac{k+l}{2}-1, \frac{k+l}{2}]$ is called the critical strip for nontrivial zeroes. Since we will prove that $L_{f\otimes g}(s,\chi)$ has no zero on $\operatorname{Re} s = \frac{k+l}{2}$, the critical strip can also be defined as $\left(\frac{k+l}{2}-1, \frac{k+l}{2}\right)$.

Proof of zero-free regions of $L_{f\otimes g}(s,\chi)$. Using the Euler product, we have

$$\log L_{f\otimes g}(s,\chi) = -\sum_p \left(\log\left(1-\alpha_f(p)\alpha_g(p)\chi(p)p^{-s}\right)+\log\left(1-\bar{\alpha}_f(p)\bar{\alpha}_g(p)\chi(p)p^{-s}\right)\right.$$
$$\left.+\log\left(1-\bar{\alpha}_f(p)\alpha_g(p)\chi(p)p^{-s}\right)+\log\left(1-\alpha_f(p)\bar{\alpha}_g(p)\chi(p)p^{-s}\right)\right)$$
$$= \sum_p\sum_m \frac{\left(\alpha_f^m(p)+\bar{\alpha}_f^m(p)\right)\left(\alpha_g^m(p)+\bar{\alpha}_g^m(p)\right)\chi(p^m)}{mp^{ms}}$$
$$=: \sum_p\sum_m \frac{\mu_f(p^m)\mu_g(p^m)\chi(p^m)}{mp^{ms}} \quad \text{say,}$$

where $\mu_f(p^m) := \left(\alpha_f^m(p) + \bar{\alpha}_f^m(p)\right)$must be real. Hence,

$$-\frac{L'_{f\otimes g}(s,\chi)}{L_{f\otimes g}(s,\chi)} = \sum_{n=1}^{\infty} \frac{\chi(n)\,\mu_f(n)\,\mu_g(n)\,\Lambda(n)}{n^s}, \tag{26}$$

where the von-Mangoldt function is defined as $\Lambda(n) = \log p$ if $n = p^m$ and 0 otherwise.

If $f \neq g$, let

$$H(s,\chi) = L_{f\otimes f}(s+k-1,\chi)\,L_{g\otimes g}(s+l-1,\chi)\,L^2_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi\right).$$

We see that

$$-\frac{H'(s,\chi)}{H(s,\chi)} = \sum_{n=1}^{\infty} \frac{\chi(n)\left(\mu_f(n)\,n^{\frac{l-k}{4}} + \mu_g(n)\,n^{\frac{k-l}{4}}\right)^2 \Lambda(n)}{n^{\left(s+\frac{k+l}{2}-1\right)}}. \tag{27}$$

We write

$$\phi(w) = H^6(w,\chi_0)\,H^4(w+it,\chi)\,H^4(w-it,\bar{\chi})\,H\left(w+2it,\chi^2\right)H\left(w-2it,\bar{\chi}^2\right).$$

Obviously,

$$\begin{aligned}-\frac{\phi'(w)}{\phi(w)} &= 2\sum_{n=1}^{\infty} \frac{\left(\mu_f(n)\,n^{\frac{l-k}{4}} + \mu_g(n)\,n^{\frac{k-l}{4}}\right)^2 \Lambda(n)}{n^{\left(w+\frac{k+l}{2}-1\right)}}\chi_0(n) \\ &\quad\times\left\{3+4\cos\left(t\log n+\arg\chi(n)\right)+\cos\left(2\left(t\log n+\arg\chi(n)\right)\right)\right\},\end{aligned}$$

which proves that $\phi(w)$ satisfies the positivity condition of Lemma 2. Let χ_2^* be the primitive character which induce χ^2 and set

$$\Phi(w) = \phi(w)\prod_{p|q} P^{-1}\left(\chi_2^*(p)\,p^{-(w+2it)}\right)P^{-1}\left(\bar{\chi}_2^*(p)\,p^{-(w-2it)}\right) \tag{28}$$

where

$$\begin{aligned}P(x) &= \left(1-\alpha_f^2(p)\,p^{-(k-1)}x\right)\left(1-\bar{\alpha}_f^2(p)\,p^{-(k-1)}x\right)\left(1-\bar{\alpha}_f(p)\,\alpha_f(p)\,p^{-(k-1)}x\right)^2 \\ &\quad\times\left(1-\alpha_g^2(p)\,p^{-(l-1)}x\right)\left(1-\bar{\alpha}_g^2(p)\,p^{-(l-1)}x\right)\left(1-\bar{\alpha}_g(p)\,\alpha_g(p)\,p^{-(l-1)}x\right)^2 \\ &\quad\times\left(1-\alpha_f(p)\,\alpha_g(p)\,p^{-\left(\frac{k+l}{2}-1\right)}x\right)^2\left(1-\bar{\alpha}_f(p)\,\bar{\alpha}_g(p)\,p^{-\left(\frac{k+l}{2}-1\right)}x\right)^2 \\ &\quad\times\left(1-\bar{\alpha}_f(p)\,\alpha_g(p)\,p^{-\left(\frac{k+l}{2}-1\right)}x\right)^2\left(1-\alpha_f(p)\,\bar{\alpha}_g(p)\,p^{-\left(\frac{k+l}{2}-1\right)}x\right)^2.\end{aligned}$$

By Ramanujan conjecture proved by P. Delign, $|\alpha_f(p)| = p^{\frac{k-1}{2}}, |\alpha_g(p)| = p^{\frac{l-1}{2}}$, we see that the function $P\left(\chi^*(p)\,p^{-(w+2it)}\right)P\left(\bar{\chi}_2^*(p)\,p^{-(w-2it)}\right)$ will only have

zeroes at $\operatorname{Re} w = 0$. Hence $\Phi(w)$ has the same zeroes with $\phi(w)$ in the critical strip $\operatorname{Re} s \in (0,1]$. In view of the functional equation for primitive characters χ (25) and formula (24), $\Phi(w)$ satisfies a desired functional equation in Lemma 2 with analytic conductor as

$$M = q(|t| + k + l + 2) \text{ and } M_0 = q(k + l + 2).$$

Though $\Phi(w)$ may be not a positive Dirichlet series, but for $\operatorname{Re} w > 1$, we have

$$\begin{aligned}\frac{\Phi'}{\Phi}(w) &= \frac{\phi'}{\phi}(w) + \sum_{p|q}\left\{-\frac{\mathrm{d}\left(\log P\left(\chi^*(p)\, p^{-(w+2it)}\right)\right)}{\mathrm{d}\, w}\right.\\ &\quad \left.-\frac{\mathrm{d}\left(\log P\left(\bar{\chi}^*(p)\, p^{-(w-2it)}\right)\right)}{\mathrm{d}\, w}\right\}\\ &=: \frac{\phi'}{\phi}(w) + T(w),\end{aligned}$$

where

$$|T(w)| \leq 32 \sum_{p|q} \frac{p^{-\operatorname{Re} w} \log p}{1 - p^{-\operatorname{Re} w}} \leq 32 \log q.$$

This shows that formula (11) still holds for $\Phi(w)$, and then Lemma 2 also follows for such $\Phi(w)$.

We first consider the case $\chi^2 \neq \chi_0$. Then $\Phi(w)$ has a pole of multiplicity 12 at $w = 1$ and $\kappa = 0$. Suppose that $L_{f\otimes g}\left(s + \frac{k+l}{2} - 1, \chi\right) = L_{f\otimes g}\left(w + it + \frac{k+l}{2} - 1, \chi\right) = 0$, $\operatorname{Re} s = w$ in the region

$$1 - \frac{c}{\log q(|t| + k + l + 2)} < w \leq 1.$$

Thus, $\Phi(w)$ has at least 16 real zeroes which is a contradiction to Lemma 1 (4).

Second, if $\chi^2 = \chi_0$, then $\Phi(w)$ has 12 poles at $w = 1$ and $\kappa = 2$. If $|t| > 4c_1^{\frac{1}{4}}/\log M_0$, suppose $L_{f\otimes g}\left(s + \frac{k+l}{2} - 1, \chi\right) = L_{f\otimes g}\left(w + it + \frac{k+l}{2} - 1, \chi\right) = 0$, $\operatorname{Re} s = w$, in the region

$$1 - \frac{c_1}{\log q(|t| + k + l + 2)} < w \leq 1.$$

Then $\Phi(w)$ has at least 16 real zeroes which contradicts to Lemma 2 (7). For the complex zeroes with imaginary part of small absolute value, we suppose $L_{f\otimes g}\left(w + \frac{k+l}{2} - 1, \chi\right)$ has a complex zero in the region

$$1 - \frac{c_2}{\log q(k + l + 2)} < \operatorname{Re} w \leq 1, \quad |t| = |\operatorname{Im} w| < \frac{c_2^{\frac{1}{2}}}{2\log M_0}.$$

Taking $t = 0$, $\Phi(w)$ has at least 16 pairs of conjugate complex zeroes but 16 poles at $w = 1$ and $\kappa = 0$ which is a contradiction to Lemma 2 (8). Choosing

$4c_1^{\frac{1}{4}} < c_2^{\frac{1}{2}}/2$, we find that $L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi\right)$ has no complex zeroes in the region

$$1-\frac{c}{\log q\left(|t|+k+l+2\right)}<\operatorname{Re}s\leq 1. \tag{29}$$

$\Phi(w)$ has at most one simple real zero. Otherwise, $\Phi(w)$ would have at least 32 real zeroes which contradicts Lemma 1 (4).

If $f=g$, $H(s,\chi)=L_{f\otimes f}^4(s+k-1,\chi)$. In the case that $\chi^2\neq\chi_0$, $\Phi(w)$ has 24 poles at $w=1$ and $\kappa=0$, from which we derive as above that $L_{f\otimes f}(s+k-1,\chi)$ has no zero in the region (29) for $k=l$. If $\chi^2=\chi_0$, $\chi\neq\chi_0$, $\Phi(w)$ has 24 poles at $w=1$ and $\kappa=4$, and by a similar argument as before, $L_{f\otimes f}(s+k-1,\chi)$ has no complex zero in the region (29) for $k=l$. If a real zero exists in this region, it must be single and simple. If χ is the principal character, we take $\phi(s)=L_{f\otimes f}(s+k-1,\chi_0)$ which is a positive Dirichlet series and has a simple pole at $s=1$. Thus, by Lemma 1 for the corresponding $\Phi(s)$ defined similar to formula (28), $L_{f\otimes f}(s+k-1,\chi_0)$ has no complex zero in the region (29), and it has at most one simple real zero in this region.

Further, we assume that χ_1 and χ_2 are two distinct primitive real characters modulo q_1, $q_2\geq 2$ respectively, and let β_1 and β_2 be the real zeroes of $L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi_1\right)$ and $L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi_2\right)$ in the region

$$1-\frac{c}{\log q_1q_2\left(k+l+2\right)}<s\leq 1. \tag{30}$$

We set

$$\phi(s)=H(s,1)\,H(s,\chi_1)\,H(s,\chi_2)\,H(s,\chi_1\chi_2),$$

and see from (27) that

$$-\frac{\phi'(s)}{\phi(s)}=\sum\frac{\Lambda(n)}{n^{-(s+\frac{k+l}{2}-1)}}\left(1+\chi_1(n)\right)\left(1+\chi_2(n)\right)\left(\mu_f(n)\,n^{\frac{l-k}{4}}+\mu_g(n)\,n^{\frac{k-l}{4}}\right)^2.$$

$\phi(s)$ is then a positive L-series which satisfies the positivity condition and the desired functional equation. By our assumption, $\phi(s)$ has at least 4 zeroes in the region (30), but only 2 poles at $s=1$ and $\kappa=0$ which is a contradiction to Lemma 1. Hence we must have

$$\min\left(\beta_1,\beta_2\right)<1-\frac{c}{\log q_1q_2\left(k+l+2\right)}.$$

■

We summarize the results of this section in the following Theorem:

Theorem 3 *Let $s=\sigma+it$, χ be a Dirichlet character modulo q, $f\in S_k(\Gamma)$ and $g\in S_l(\Gamma)$ are normalized Hecke eigenforms. Then there exists an absolute constant $c>0$ such that the following holds:*

1. If χ is a complex character, $L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi\right)$ has no zeroes in the region
$$\sigma\geq 1-\frac{c}{\log q\left(|t|+k+l+2\right)}.$$

2. If χ is a real character, $L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi\right)$ has no complex zeroes in the region
$$\sigma \geq 1-\frac{c}{\log q\left(|t|+k+l+2\right)},$$
and at most one simple real zeroes exists in this region.

3. If χ_1 and χ_2 are two distinct real primitive characters modulo q_1 and $q_2 \geq 2$ respectively, and β_1 and β_2 are real zeroes of $L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi_1\right)$ and $L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi_2\right)$, then
$$\min\left(\beta_1,\beta_2\right) < 1-\frac{c}{\log q_1 q_2\left(k+l+2\right)}.$$

Following the argument in [1, p. 130], statement 3 implies the following:

(a) There is at most one character χ modulo q such that the corresponding function $L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi\right)$ has a real, simple zero β satisfying
$$1>\beta \geq 1-\frac{c}{\log q\left(k+l+2\right)}.$$

(b) For $2 \leq q \leq x$, there exists at most one q_1 with $2 \leq q_1 \leq x$ and at most one real primitive character χ_1 modulo q_1 such that $L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi_1\right)$ has a real simple zero β_1 satisfying
$$1>\beta_1 \geq 1-\frac{c}{\log x\left(k+l+2\right)}.$$

Moreover, if χ is a real character modulo q and $L_{f\otimes g}\left(s+\frac{k+l}{2}-1,\chi\right)$ has real zeroes in the above range, then $q \equiv 0 \left(\operatorname{mod} q_1\right)$.

Remark 3 *Theorem 3 implies the results of Perreli [18, Lemma 3, 4] for $f=g$, and of Ichihara [19, Theorem 1] for $f \neq g$. In contrast to their results, our constant c does not depends on the weights k and l.*

Remark 4 *If π and π' are the unitary cuspidal automorphic representation of $GL_m\left(\mathbb{A}_K\right)$ and $GL_{m'}\left(\mathbb{A}_K\right)$. For $m=m'=2$, Moreno [20] [21] has proved the logarithmic zero-free regions for the two cases that π and π' are either twisted equivalent by $||^{i\tau_0}$, or that π and π' are not twisted equivalent by $||^{i\tau}$ but π,π' must be both self-dual. For the general m,m', if π,π' are not twisted equivalent by $||^{i\tau}$, and π is self-dual (self-contragredient) whereas π' is not, then $L\left(s,\pi\times\pi'\right)$ can be proved to have a logarithmic zero-free region and not to have Siegel zero by applying Lemma 1 of [4] to $L\left(s,\Pi\times\Pi\right)$, where Π is the isobaric automorphic representation*
$$\Pi=\pi\boxplus\left(\pi'\otimes\alpha^{-it}\right)\boxplus\left(\tilde{\pi}'\otimes\alpha^{it}\right).$$

For the case that π and π' are twisted equivalent by $||^{i\tau_0}$, and the case that π and π' are not twisted equivalent by $||^{i\tau}$ but π, π' are both self-dual, Sarnak [7] introduce the isobaric automorphic representation

$$\Pi = \pi \boxplus \left(\pi \otimes \alpha^{it}\right) \boxplus \left(\pi \otimes \alpha^{-it}\right) \boxplus \pi' \boxplus \left(\pi' \otimes \alpha^{it}\right) \boxplus \left(\pi' \otimes \alpha^{-it}\right)$$

to construct a positive Dirichlet series $L\left(w, \Pi \times \tilde{\Pi}\right)$. But $L\left(w, \Pi \times \tilde{\Pi}\right)$ may have other poles besides $w = 1$, so for obtaining the logarithmic zero-free region, it is not sufficient to apply only Lemma 1 as in [4, Lemma 1]. Our Lemma 2 is then generated to deal with such $L\left(w, \Pi \times \tilde{\Pi}\right)$ for getting the logarithmic zero-free region of $L(s, \pi \times \pi')$.

Acknowledgement The authors would like to thank Tonghai Yang, E. Lapid and Zhenyu Mao for very helpful conversations. The authors also thank Jianya Liu and Yangbo Ye for several invitations and stimulating discussions at Shandong University.

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